

Apolarity for determinants and permanents of generic symmetric matrices

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March 11, 2013

Abstract

We show that the apolar ideal to the determinant of a generic symmetric matrix is generated in degree two, and the apolar ideal to the permanent of a generic symmetric matrix is generated in degrees two and three. In each case we specify the generators of the apolar ideal. As a consequence, using a result of K. Ranestad and F. O. Schreyer we give lower bounds to the cactus rank and rank of each of these polynomials. We compare these bounds with those obtained by J. Landsberg and Z. Teitler.

1 Introduction

This paper is a sequel to [Sh]. In this paper we determine the annihilator ideals of the determinant and the permanent of a generic symmetric $n \times n$ matrix. Here annihilator is meant in the sense of the apolar pairing, i.e. Macaulay's inverse system. In section two we review the doset basis of the space of $t \times t$ minors of an $n \times n$ symmetric matrix. In section three we determine the generators of the apolar ideal to the determinant and permanent of a generic symmetric matrix (Theorems 3.11 and 3.22). In section four we apply our result to find a lower bound for the scheme/cactus rank of the determinant and permanent of the generic symmetric matrix (Theorems 4.4 and 4.5).

Let k be an infinite field of characteristic zero or characteristic $p > 2$, and $X = (x_{ij})$ be a square symmetric matrix of size n with $\frac{n(n+1)}{2}$ distinct variables. The determinant and the permanent of X are polynomials of degree n . Let $R^s = k[x_{ij}](i \leq j)$, be a polynomial ring and $S^s = k[y_{ij}](i \leq j)$, be the ring of differential operators associated to R^s , and let R_k^s and S_k^s denote the degree- k homogeneous summands. S^s acts on R^s by differentiation.

Note that the determinant and permanent of the symmetric matrices contain squares of

some variables. Hence it makes a difference when one writes the determinant of a symmetric matrix in a divided power ring or in the usual polynomial ring. We can choose to write the determinant in the usual polynomial ring and use the differentiation to find the apolar ideal or write the determinant in the divided powers ring (Definition 1.1 below) and use the contraction.

For the main results in the section three we use the usual polynomial ring and the differentiation unless otherwise stated. In section five we consider what happens when we write the determinant in the usual polynomial ring but find the apolar ideals using the contraction instead of differentiation.

Definition 1.1. ([IK], Appendix A) Let k be a field of arbitrary characteristic. Let $R = k[x_1, \dots, x_r] = \bigoplus_{j \geq 0} R_j$, Let \mathcal{D} be the graded dual of R , i.e.

$$\mathcal{D} = \bigoplus_{j \geq 0} \text{Hom}_k(R_j, k) = \bigoplus_{j \geq 0} \mathcal{D}_j.$$

We consider the vector space R_1 with the basis x_1, \dots, x_r and the left action of $GL_r(k)$ on R_1 defined by $Ax_i = \sum_{j=1}^r A_{ij}x_j$. Since $R = \bigoplus_{j \geq 0} \text{Sym}^j R_1$ this action extends to an action of $GL_r(k)$ on R . By duality this action determines a left action of $GL_r(k)$ on $\bigoplus_{j \geq 0} \mathcal{D}_j$. We denote by $x^U = x_1^{u_1} \dots x_r^{u_r}$, $|U| = u_1 + \dots + u_r = j$ the standard monomial basis of R_j . Let X_1, \dots, X_r be the basis of \mathcal{D}_1 dual to the basis x_1, \dots, x_r . We denote by

$$X^U = X_1^{[u_1]} \dots X_r^{[u_r]}$$

the basis of \mathcal{D}_j dual to the basis $\{x^U : |U| = j\}$. We call these elements divided power monomials. We call the elements of \mathcal{D}_j divided power forms, and the elements of \mathcal{D} divided power polynomials. We extend the definition of $X^{[U]}$ to multi-degrees $U = (u_1, \dots, u_r)$ with negative components by letting $X^{[U]} = 0$ if $u_i < 0$ for some i .

We define a ring structure on \mathcal{D} . One defines multiplication of monomials by the equality

$$X^{[U]} \cdot X^{[V]} = \binom{U+V}{U} X^{[U+V]},$$

where $\binom{U+V}{U}$ is a product of binomial coefficients. This is extended by linearity and gives \mathcal{D} a structure of a k -algebra.

Notation. Throughout this section we let $X = (x_{ij})$ with $x_{ij} = x_{ji}$ be an $n \times n$ symmetric matrix of indeterminates in the polynomial ring $R^s = k[x_{ij}]$. Let \mathcal{D} be the corresponding divided power ring. Let $Y = (y_{ij})$ with $y_{ij} = y_{ji}$ be an $n \times n$ symmetric matrix of indeterminates in the ring of inverse polynomials $S^s = k[y_{ij}]$ associated to R^s .

To each degree- j homogeneous element, $F \in R_j^s$ we can associate the ideal $I = \text{Ann}(F)$ in $S^s = \mathbb{k}[y_{ij}]$ consisting of polynomials Φ such that $\Phi \circ F = 0$. We call $I = \text{Ann}(F)$, the apolar ideal of F ; and the quotient algebra $S^s/\text{Ann}(F)$ the apolar algebra of F . If $h \in S_k^s$ and $F \in R_n^s$, then we have $h \circ F \in R_{n-k}^s$.

Let $F \in R^s$, then $\text{Ann}(F) \subset S^s$ and we have

$$(\text{Ann}(F))_k = \{h \in S_k^s \mid h \circ F = 0\}.$$

Let $\phi : (S_i^s, R_i^s) \rightarrow \mathbb{k}$ be the pairing $\phi(g, f) = g \circ f$, and V be a vector subspace of R_k , then we have

$$\dim_{\mathbb{k}}(V^\perp) = \dim_{\mathbb{k}} S_k^s - \dim_{\mathbb{k}} V.$$

For $V \subset R_k^s$, we denote by $V^\perp = \text{Ann}(V) \cap S_k^s$.

Let F be a form of degree j in R^s . We denote by $\langle F \rangle_{j-k}$ the vector space $S_k^s \circ F \subset R_{j-k}^s$.

Remark 1.2. (see [IK]) Let $F \in R^s$ and $\deg F = j$ and $k \leq j$. Then we have

$$(\text{Ann}(F))_k = \{h \in S_k^s \mid h \circ S^{s^{j-k}} F = 0\} = (\text{Ann}(S^{s^{j-k}} F))_k.$$

We define the homomorphism $\xi : R^s \rightarrow S^s$ by setting $\xi(x_{ij}) = y_{ij}$; for a monomial $v \in R^s$ we denote by $\hat{v} = \xi(v)$ the corresponding monomial of S^s .

Remark 1.3. (see [Sh], Remark 2.8) Let $f = \sum_{i=1}^{i=k} \alpha_i v_i \in R_n^s$ with $\alpha_i \in \mathbb{k}$ and with v_i 's linearly independent monomials. Then we will have:

$$\text{Ann}(f) \cap S_n^s = \langle \alpha_j \hat{v}_1 - \alpha_1 \hat{v}_j, \langle v_1, \dots, v_k \rangle^\perp \rangle,$$

where $\langle v_1, \dots, v_k \rangle^\perp = \text{Ann}(\langle v_1, \dots, v_k \rangle) \cap S_n^s$.

Denote by $\mathfrak{A}_X = S^s/(\text{Ann}(\det(X)))$ the *apolar algebra* of the determinant of the matrix X . Recall that the Hilbert function of \mathfrak{A}_X is defined by $H(\mathfrak{A}_X)_i = \dim_{\mathbb{k}}(\mathfrak{A}_X)_i$ for all $i = 0, 1, \dots$.

Definition 1.4. Let F be a polynomial in R^s , we define the $\deg(\text{Ann}(F))$ to be the length of $S^s/\text{Ann}(F)$.

1.1 Summary of main results

- We specify the Hilbert sequence corresponding to the apolar algebra of the following homogeneous polynomials
 - Determinant of a generic symmetric matrix (Table 1). This uses the Conca Theorem 2.2.
 - Permanent of the generic symmetric matrix (Table 2).
- We specify the generators of the apolar ideal in each of the following cases:
 - Determinant of a generic symmetric $n \times n$ matrix. This ideal is generated by certain 2×2 permanents, certain degree two trinomials that are Hafnians of 4×4 symmetric submatrices and some monomials (Proposition 3.1). In particular, this ideal is generated in degree two (Theorem 3.11).
 - Permanent of a generic symmetric $n \times n$ matrix. This ideal is generated by certain 2×2 minors, certain degree three polynomials corresponding to 6×6 Hafnians and some degree two monomials (Proposition 3.13, Lemma 3.18). In particular, this ideal is generated in degrees two and three (Theorem 3.23).
- In each of the above cases the proof has several main steps:
 - a. Identify the dual module to S/I , so determine $S_i \circ F$, where F is the invariant.
 - b. For the determinant we determine I_2 , where I is the apolar ideal; and for the permanent we determine I_2 and I_3 , and let $I^+ = (I_2, I_3)$.
 - c. In the case of the determinant we show that $(I_2)_k$ is the full perpendicular space in S_k to $S_{n-k} \circ \det(X)$. And in the case of permanent we show that $(I^+)_k$ is the full perpendicular space in S_k to $S_{n-k} \circ \text{perm}(X)$.

Of these steps, the last is the hardest and we use a triangularity method. For the determinant of a symmetric matrix we show that the acceptable monomials which are not the leading term of a Conca doset minor are the initial monomial of the generators of the ideal (I_2) in the reverse lexicographic order (Proposition 3.8), and for the permanent of symmetric matrix we use a similar triangularity method (Proposition 3.19).
- We apply these results to give a lower bound for:
 - Cactus rank of the determinant of a generic symmetric $n \times n$ matrix (Theorem 4.4).
 - Rank of the determinant of the generic symmetric $n \times n$ matrix (Proposition 4.7).

- Cactus rank of the permanent of a generic symmetric $n \times n$ matrix (Theorem 4.5).
- We give a Gröbner basis for the apolar ideal of the determinant of a generic symmetric matrix (Theorem 3.12).

Acknowledgment. *I am deeply grateful to my advisor Prof. A. Iarrobino whose help, stimulating ideas and encouragement helped me in working on this problem and writing this paper. I am very thankful to Prof. Z. Teitler for suggesting this problem and his helpful comments and also Prof. A. Conca and Prof. L. Smith for their valuable comments and suggestions. I also gratefully acknowledge support in summer 2012 from the Ling-Ma fellowship of the Department of Mathematics at Northeastern University.*

2 Doset basis for the space of $k \times k$ minors

We recall the definition of doset minors and the Gröbner basis for the determinantal ideal of a generic symmetric matrix.

Definition 2.1. (see [CON]) Let H be the set of all subsequences (a_1, \dots, a_t) of $(1, \dots, n)$. Let $a, b \in H$. We define on H the partial order

$$a = (a_1, \dots, a_t) \leq b = (b_1, \dots, b_r) \iff r \leq t,$$

and

$$a_i \leq b_i \text{ for } i = 1, \dots, r.$$

We denote by $[a_1, \dots, a_t | b_1, \dots, b_t]$ the minor $\det(X_{a_i b_j}), 1 \leq i, j \leq t$ of X . Since X is symmetric it is clear that $[a, b] = [b, a]$. A minor $[a_1, \dots, a_t | b_1, \dots, b_t]$ of X is called a doset minor if $a \leq b$ in H .

Let τ be a diagonal term order on $R^s = \mathbb{k}[x_{ij}]$ such that the initial term of every doset minor $[a_1, \dots, a_s | b_1, \dots, b_s]$ is $\prod_{i=1}^s x_{a_i b_i}$. For instance, we can consider lexicographic order induced by the variable order

$$x_{11} \geq x_{12} \geq \dots \geq x_{1n} \geq x_{22} \geq \dots \geq x_{2n} \geq \dots \geq x_{n-1n} \geq x_{nn}.$$

Theorem 2.2. (Conca)[[CON], Theorem 2.9] *Let $I_t(X)$ be the ideal generated by the t -minors of X . The set of the doset t -minors is a Gröbner basis for $I_t(X)$ with respect to τ .*

Definition 2.3. A Young tableaux of shape (r_1, \dots, r_u) is an array of positive integers $A = (a_{ij})$, $1 \leq i \leq u$ and $1 \leq j \leq r_i$ with $r_1 \geq \dots \geq r_u$. Such a tableaux is said to be semi-standard if it has the numbers in each row in strictly increasing order from left to right, and the numbers in each column are in non-decreasing order from top to bottom (that is $a_{i,j} < a_{i,j+1}$ for all $i = 1, \dots, u, j = 1, \dots, r_i - 1$ and $a_{i,j} \leq a_{i+1,j}$ for all $i = 1, \dots, u - 1, j = 1, \dots, r_{i+1}$).

Example. An example of a semi-standard Young Tableaux of shape $(4, 3, 2, 2, 1)$ filled with the numbers $\{1, 2, 3, 4\}$ is

1	2	3	4
1	3	4	
2	3		
2	4		
2			

Definition 2.4. A Dyck path of order n is a diagonal avoiding staircase walk or path with t interior vertices $(0, 0) \rightarrow (a_1, b_1) \rightarrow \dots \rightarrow (a_t, b_t) \rightarrow (n, n)$ from $(0, 0)$ to (n, n) .

The number of Dyck paths of order n is given by the Catalan number ([ST], Volume 2, page 205)

$$c_n = \frac{1}{n+1} \binom{2n}{n}.$$

Lemma 2.5. *The dimension of the space of $t \times t$ minors of an $n \times n$ symmetric matrix is equal to the number of the doset t -minors of the $n \times n$ symmetric matrix. This is equal to the number of fillings of a semi-standard Young tableaux of shape (t, t) with the numbers $\{1, \dots, n\}$, which is equal to the Narayana number*

$$N(n, t) = \binom{n+1}{t} \binom{n+1}{t+1} / (n+1).$$

Proof. The first statement is true by Conca's Theorem (Theorem 2.2). To show the second statement one can view the count of the Conca doset t -minors as giving the coordinates in the x - y plane of t points. Then the count is of segmented paths with t interior vertices lying on or above the diagonal, beginning at $(0, 0)$ and ending at $(n+1, n+1)$. That is the number of Dyck n -paths with exactly t vertices, which is given by the Narayana numbers as $\binom{n+1}{t} \binom{n+1}{t+1} / (n+1)$ (See [ST], Volume 2, page 237, Exercise 6.36).

□

Corollary 2.6. *Let X be a generic symmetric $n \times n$ matrix, then $\deg(\text{Ann}(\det(X)))$ is the Catalan number $C_{n+1} = \frac{1}{n+2} \binom{2n+2}{n+1}$.*

Proof. Let $\langle I_t \rangle$ be the space of $t \times t$ minors of a symmetric $n \times n$ matrix, then we have:

$$\deg(\text{Ann}(\det(X))) = \sum_{t=0}^{t=n} \dim(\langle I_t \rangle).$$

Thus the $\deg(\text{Ann}(\det(A)))$ will be the total number of diagonal-avoiding paths through the $n \times n$ grid, which is given by the Catalan number $C_{n+1} = \frac{1}{n+2} \binom{2n+2}{n+1}$. (See [ST], Volume 2, page 173, Corollary 6.2.3)

□

Table 1: The Hilbert sequence of the determinant of the generic symmetric matrix

n=2	1	3	1						
n=3	1	6	6	1					
n=4	1	10	20	10	1				
n=5	1	15	50	50	15	1			
n=6	1	21	105	175	105	21	1		
n=7	1	28	196	490	490	196	28	1	
n=8	1	36	336	1176	1764	1176	336	36	1

3 Generators of the apolar ideal

In section 3.1 we determine the generators of the apolar ideal of the determinant of the $n \times n$ generic symmetric matrix. In section 3.2 we determine the generators of the apolar ideal of the permanent of the $n \times n$ generic symmetric matrix.

Notation. ([IKO]) Let $F_{2m} \subset S_{2m}$ be the set of all permutations σ satisfying the following conditions:

- (1) $\sigma(1) < \sigma(3) < \dots < \sigma(2m-1)$
- (2) $\sigma(2i-1) < \sigma(2i)$ for all $1 \leq i \leq m$

We denote by $Hf(X)$ the Hafnian of a generic symmetric $2n \times 2n$ matrix X , defined by

$$Hf(X) = \sum_{\sigma \in F_{2n}} x_{\sigma(1)\sigma(2)} x_{\sigma(3)\sigma(4)} \dots x_{\sigma(2n-1)\sigma(2n)} \quad (1)$$

3.1 Apolar ideal of the determinant

In this subsection we determine the apolar ideal of the determinant of the $n \times n$ generic symmetric matrix, and we will show that it is generated by degree two. We first determine the generators in degree two (Proposition 3.1). We then show that the generators in degree two generate the annihilator ideal in degree n (Proposition 3.6). Then we show that the degree two generators generate the ideal in each degree k for $2 \leq k \leq n$. A key step is to use triangularity to show that these degree two generators, generate all the apolar ideal (Lemma 3.7 and Proposition 3.8). This leads to our main result (Theorem 3.11).

Notation. For the generic symmetric $n \times n$ matrix X , the unacceptable monomials of degree k in S_k^s are monomials which do not divide any term of the determinant of X . We denote the set of degree k unacceptable monomials by U_k . A monomial that divides some term of the determinant is called an acceptable monomial.

Proposition 3.1. *For an $n \times n$ symmetric matrix $X = (x_{ij})$, $\text{Ann}(\det(X)) \subset S^s$, includes the following degree 2 polynomials:*

- (a) *Unacceptable monomials including $y_{ii}y_{ij}$ for all $1 \leq i, j \leq n$. The number of these monomials are n^2 .*
- (b) *All the diagonal 2×2 binomials of the form $y_{ij}^2 + 2y_{ii}y_{jj}$. The number of these binomials is $\binom{n}{2}$.*
- (c) *All the 2×2 permanents with one diagonal element, i.e. $y_{jk}y_{il} + y_{jl}y_{ii}$. The number of these binomials is $n \cdot \binom{n-1}{2}$.*
- (d) *The Hafnians of all symmetrically chosen 4×4 submatrices of X . The number of these trinomials is $\binom{n}{4}$.*

Proof. We have $\det(X) = \sum_{\sigma \in S_n} \text{Sgn}(\sigma) \prod x_{i, \sigma(i)}$. First we show that monomials of type (a) are in $\text{Ann}(\det(X))$. By symmetry we have

$$y_{ii}y_{ij} \circ \det(X) = 0 \text{ (where } j \geq i \text{),}$$

$$y_{ii}y_{ji} \circ \det(X) = 0 \text{ (where } j \leq i \text{).}$$

Now we want to show that binomials of type (b) are in $\text{Ann}(\det(X))$. Let $P = 2y_{ii}y_{jj} + y_{ij}^2$.

There are $n!$ terms in the expansion of the determinant. If a term doesn't contain the monomial $x_{ii}x_{jj}$ or the monomial x_{ij}^2 then the result of action of P on it will be zero. Let σ_1 be a permutation having x_{ii} and x_{jj} respectively in its i -th and j -th place. Corresponding to σ_1 we also have a permutation $\sigma_2 = \tau\sigma_1$, where $\tau = (i, j)$ is a transposition and $\text{sgn}(\sigma_2) =$

$\text{sgn}(\tau\sigma_1) = -\text{sgn}(\sigma_1)$. Thus, corresponding to each positive term in the determinant which contains the monomial $x_{ii}x_{jj}$ or the monomial x_{ij}^2 we have the same term with the negative sign, thus the resulting action of the binomial P on $\det(X)$ is zero.

To show that the binomials of type (c) are in the annihilator ideal we can use the same proof as we used for the binomials of type (b).

Now we want to show that any 4×4 Hafnian of Y annihilates the determinant of an $n \times n$ symmetric matrix X . For $n = 4$ it is easy to check that the 4×4 Hafnian annihilates the determinant of X .

Now let $n \geq 4$. Let W be a 4×4 submatrix of Y , involving the rows and the columns i_1, i_2, i_3 and i_4 .

$$W = \begin{pmatrix} y_{i_1 i_1} & y_{i_1 i_2} & y_{i_1 i_3} & y_{i_1 i_4} \\ y_{i_2 i_1} & y_{i_2 i_2} & y_{i_2 i_3} & y_{i_2 i_4} \\ y_{i_3 i_1} & y_{i_3 i_2} & y_{i_3 i_3} & y_{i_3 i_4} \\ y_{i_4 i_1} & y_{i_4 i_2} & y_{i_4 i_3} & y_{i_4 i_4} \end{pmatrix}$$

Using Equation (1) the Hafnian of W is

$$H = \text{Hf}(W) = y_{i_1 i_2} y_{i_3 i_4} + y_{i_1 i_3} y_{i_2 i_4} + y_{i_1 i_4} y_{i_2 i_3}.$$

If a term in the determinant does not contain the monomials $x_{i_1 i_2} x_{i_3 i_4}$ or $x_{i_1 i_3} x_{i_2 i_4}$ or $x_{i_1 i_4} x_{i_2 i_3}$, then H annihilates it. If a term in the determinant contains one of the monomials $x_{i_1 i_2} x_{i_3 i_4}$ or $x_{i_1 i_3} x_{i_2 i_4}$ or $x_{i_1 i_4} x_{i_2 i_3}$, since these monomials do not appear in any other 4×4 sub matrix of X , we can use the Laplace expansion (cofactor expansion) of the determinant and the proof is complete. □

We denote by $\{V\}$ be set of the degree two elements of type (a), (b), (c) and (d) in Proposition 4.4, and by V the vector subspace of S^s spanned by $\{V\}$. We denote by $\{a\}$, $\{b\}$, $\{c\}$ and $\{d\}$ the set of elements in (a), (b), (c) and (d) respectively.

Lemma 3.2. *The set $\{V\}$ is linearly independent and we have,*

$$\dim V = n^2 + \binom{n}{2} + n \cdot \binom{n-1}{2} + \binom{n}{4}.$$

Proof. Each of the four subsets are linearly independent from each other since they involve different variables. So it suffices to show that each subset is linearly independent. The

subset $\{a\}$ is linearly independent since the monomials in $\{a\}$ form a Gröbner basis for the ideal they generate. The subsets $\{b\}$ and $\{c\}$ are linearly independent since by choosing two elements of the matrix, where at least one element is diagonal, we have a unique 2×2 minor. The subset $\{d\}$ is linearly independent since the monomials that appear in a Hafnian of a 4×4 symmetric sub matrix of X , do not appear in the Hafnian of any other 4×4 symmetric sub matrix of X . Hence the set $\{V\}$ is linearly independent and the dimension of the vector space V is $n^2 + \binom{n}{2} + n \cdot \binom{n-1}{2} + \binom{n}{4}$.

□

Lemma 3.3.

$$S_k^s \circ (\det(X)) = M_{n-k}(X) \subset R^s. \quad (2)$$

Proof. To show the inclusion

$$S_k^s \circ (\det(X)) \subset M_{n-k}(X) \subset R^s,$$

we use induction on k . For $k = 1$, the above inclusion is easy to see. Now assume that the above inclusion holds for $k - 1$, i.e $S_{k-1}^s \circ \det(X) \subset M_{n-(k-1)}(X)$, and we want to show that it is true for k . We have

$$S_k^s \circ \det(X) = S_1^s S_{k-1}^s \circ \det(X) \subset S_1^s \circ M_{n-k+1}(X) \subset M_{n-k}(X).$$

Now we want to show the opposite inclusion

$$S_k^s \circ (\det(X)) \supset M_{n-k}(X) \subset R^s,$$

Let $M_{\hat{I}, \hat{J}}(X)$, $I = \{i_1, \dots, i_k\}$, $J = \{j_1, \dots, j_k\}$, $1 \leq i_1 \leq i_2 \leq \dots \leq i_k \leq n$, $1 \leq j_1 \leq j_2 \leq \dots \leq j_k \leq n$ be the $(n - k) \times (n - k)$ minor of X one obtains by deleting the I rows and J columns of X . Let

$$\Delta_{(I, J)} = \{(i_r, j_r) | i_r \in I, j_r \in J \text{ and } i_r = j_r\}.$$

Let $\Delta_I = \{i_r | (i_r, j_r) \in \Delta_{(I, J)}\}$ and $\Delta_J = \{j_r | (i_r, j_r) \in \Delta_{(I, J)}\}$.

Let $M_{(I, J) - \Delta}$ be the sub matrix of Y with the rows $I - \Delta_I$, and the columns $J - \Delta_J$.

We claim

$$M_{\hat{I}, \hat{J}} = \pm c \prod_{(i_r, j_r) \in \Delta_{(I, J)}} y_{i_r, j_r} \det(M_{(I, J) - \Delta}) \circ \det(X)$$

where $c \in \mathbf{k}$.

To prove the claim we use induction on $k = |I| = |J|$, the cardinality of the sets I and J . First we show the claim is true for $k = 1$. Let $I = \{i_1\}$ and $J = \{j_1\}$. We have two cases

I. $i_1 = j_1$ so $y_{i_1 j_1}$ is a diagonal element and we have

$$M_{\widehat{I}, \widehat{J}} = y_{i_1 j_1} \circ (\det(X)).$$

II. $i_1 \neq j_1$ so we have

$$y_{i_1 j_1} \circ (\det(X)) = 2M_{\widehat{I}, \widehat{J}}.$$

So for $k = 1$ the claim holds. Now assume that the claim holds for all every I and J with $|I| = |J| = k - 1$ and we want to show that the claim is also true for I and J with $|I| = |J| = k$.

Let $I = \{i_1, \dots, i_k\}$ and $J = \{j_1, \dots, j_k\}$.

Let $I' = I - \{i_1\}$ and $J' = J - \{j_1\}$. We have $|I'| = |J'| = k - 1$ so by the induction assumption we have

$$M_{\widehat{I'}, \widehat{J'}} = \pm c \prod_{(i_r, j_r) \in \Delta_{(I', J')}} y_{i_r j_r} \det(M_{(I', J') - \Delta}) \circ \det(X)$$

Now by writing the Laplace expansion of the determinant using row i_1 or column j_1 for $M_{\widehat{I}, \widehat{J}}$, we get

$$M_{\widehat{I}, \widehat{J}} = \pm c \prod_{(i_r, j_r) \in \Delta_{(I, J)}} y_{i_r j_r} \det(M_{(I, J) - \Delta}) \circ \det(X),$$

where $c \in \mathbf{k}$. Hence $M_{\widehat{I}, \widehat{J}} \in S_{n-k}^s \circ (\det(X))$.

□

Lemma 3.4. *For the generic symmetric $n \times n$ matrix X , we have*

$$V = \text{Ann}(M_2) \cap S_2^s = (\text{Ann}(\det X))_2$$

Proof. By the Lemma 3.3 we have

$$\text{Ann}(S_{n-2}^s \circ (\det(X))) \subset \text{Ann}(M_2(X)).$$

By the Proposition 3.1 we have

$$(\text{Ann}(\det(X)))_2 \supset V$$

By the Remark 1.2 we have

$$(\text{Ann}(\det(X)))_2 = (\text{Ann}(S_{n-2}^s \circ (\det(X))))_2 \subset \text{Ann}(M_2(X))$$

Hence we have

$$V \subset \text{Ann}(M_2).$$

On the other hand, using Lemma 3.2 we have

$$\dim(V) = n^2 + \binom{n}{2} + n \cdot \binom{n-1}{2} + \binom{n}{4} = \binom{\frac{n^2+n}{2} + 1}{2} - \frac{\binom{n+1}{2} \binom{n+1}{3}}{n+1} = \dim S_2^s - \dim M_2(X).$$

So we have

$$V = \text{Ann}(M_2) \cap S_2^s.$$

By the Proposition 3.1, Lemmas 3.2 and 3.3 we have

$$(\text{Ann}(\det(X)))_2 \subset \text{Ann}(M_2(X)) \subset V.$$

So we have

$$V = (\text{Ann}(\det(X)))_2.$$

□

Lemma 3.5. *Let $2 \leq k \leq n$. We have*

$$(V)_k \subset \text{Ann}(M_k(X)) \cap S_k^s. \quad (3)$$

Proof. We have

$$(A) \ V \circ \det(X) = 0 \iff V \circ S_{n-2}^s \circ (\det(X)) = 0 \iff V \circ M_2(X) = 0.$$

$$(B) \ (\text{Ann}(\det(X))) \cap S_2^s = V \Rightarrow S_{k-2}^s V \circ (S_{n-k}^s \circ \det(X)) = 0.$$

$$\Rightarrow S_{k-2}^s(V) \circ M_k(X) = 0.$$

$$\Rightarrow (V)_k \circ M_k(X) = 0. \text{ (By Remark 1.2)}$$

which proves the lemma. □

Proposition 3.6. *For $n \geq 2$ we have*

$$(V)_n = \text{Ann}(\det(X)) \cap S_n^s. \quad (4)$$

Proof. One inclusion is given by Lemma 3.5. To show the other inclusion we use induction on n . For $n = 2, 3$ the equality is easy to see. Now we want to show that equation (4) holds for $n = 4$.

$$X = (x_{ij}) = \begin{pmatrix} a & b & c & d \\ b & e & f & g \\ c & f & h & i \\ d & g & i & j \end{pmatrix},$$

$$Y = (y_{ij}) = \begin{pmatrix} A & B & C & D \\ B & E & F & G \\ C & F & H & I \\ D & G & I & J \end{pmatrix},$$

We have:

$$\det(X) = d^2 f^2 - 2cdfg + c^2 g^2 - d^2 eh + 2bdgh - ag^2 h + 2cdei - 2bdfi - 2bcgi + 2afgi + b^2 i^2 - aei^2 - c^2 ej + 2bcfj - af^2 j - b^2 hj + aehj \in R_4^s.$$

Now if we denote the determinant in the divided power ring by $\det(X)_{Div}$ we have:

$$\det(X)_{Div} = 4d^2 f^2 - 2cdfg + 4c^2 g^2 - 2d^2 eh + 2bdgh - 2ag^2 h + 2cdei - 2bdfi - 2bcgi + 2afgi + 4b^2 i^2 - 2aei^2 - 2c^2 ej + 2bcfj - 2af^2 j - 2b^2 hj + aehj \in \mathcal{D}_4.$$

We use the divided powers and the contraction in the following proof. Using the Remark 1.3, let ψ be the binomial in $\text{Ann}(\det(X)) \cap S_4^s$.

$$\psi = \alpha_\sigma(-1)^{\text{sgn}(\eta)} y_{1\eta(1)} y_{2\eta(2)} y_{3\eta(3)} y_{4\eta(4)} - \alpha_\eta(-1)^{\text{sgn}(\sigma)} y_{1\sigma(1)} y_{2\sigma(2)} y_{3\sigma(3)} y_{4\sigma(4)},$$

where $\sigma \neq \eta$ are two permutations of the set $\{1, 2, 3, 4\}$, α_η is the coefficient of the monomial $y_{1\eta(1)} y_{2\eta(2)} y_{3\eta(3)} y_{4\eta(4)}$ in $\det(X)_{Div}$ and α_σ is the coefficient of $y_{1\sigma(1)} y_{2\sigma(2)} y_{3\sigma(3)} y_{4\sigma(4)}$ in $\det(X)_{Div}$. The terms $y_{1\eta(1)} y_{2\eta(2)} y_{3\eta(3)} y_{4\eta(4)}$ and $y_{1\sigma(1)} y_{2\sigma(2)} y_{3\sigma(3)} y_{4\sigma(4)}$ cannot have 3 common factors, since if they have 3 variables in common the fourth variable is forced and it contradicts our assumption $\sigma \neq \eta$. Without loss of generality we can assume $\eta = id$. We have three different possibilities.

(i) $y_{11} y_{22} y_{33} y_{44}$ and $y_{1\sigma(1)} y_{2\sigma(2)} y_{3\sigma(3)} y_{4\sigma(4)}$ have two common factors. Without loss of generality we can assume that $\sigma(1) = 1$ and $\sigma(2) = 2$. So we have

$$\psi = 2AEHJ - (-1)^{\text{sgn}(\sigma)} AE y_{3\sigma(3)} y_{4\sigma(4)} = 2AEHJ + AEI^2 = AE(2HJ + I^2) \in (V)_4$$

(ii) $y_{11} y_{22} y_{33} y_{44}$ and $y_{1\sigma(1)} y_{2\sigma(2)} y_{3\sigma(3)} y_{4\sigma(4)}$ have only one common factor. Without loss of generality we can assume that $\sigma(1) = 1$, Since the only term in the determinant which has a and does not have e, h and j is $2afgi$ we have

$$\begin{aligned} \psi &= 2AEHJ - AFGI = 2AEHJ - AFGI + AF^2J - AF^2J = \\ &AJ(2EH + F^2) - AF(GI + FJ) \in (V)_4, \end{aligned}$$

since we know that $2EH + F^2 \in V$ and $GI + FJ \in V$.

(iii) $y_{11} y_{22} y_{33} y_{44}$ and $y_{1\sigma(1)} y_{2\sigma(2)} y_{3\sigma(3)} y_{4\sigma(4)}$ do not have any common factor. We add and subtract a term which has a common factor with $y_{11} y_{22} y_{33} y_{44}$ and a common factor with $y_{1\sigma(1)} y_{2\sigma(2)} y_{3\sigma(3)} y_{4\sigma(4)}$. The reason that such a term exists in the determinant is that if we choose two elements, α and β not in the same row or column, it is easy to see that we always have a term in the determinant containing $\alpha\beta$. On the other hand if we choose one variable from $y_{11} y_{22} y_{33} y_{44}$, say y_{11} , there is always one variable in $y_{1\sigma(1)} y_{2\sigma(2)} y_{3\sigma(3)} y_{4\sigma(4)}$ which is not in the first row or column, since we only have three elements other than y_{11} in the first row and column. So we can always choose a term in the determinant with at least one common factor with $y_{11} y_{22} y_{33} y_{44}$ and at least one common factor with $y_{1\sigma(1)} y_{2\sigma(2)} y_{3\sigma(3)} y_{4\sigma(4)}$. Then using the cases (i) or (ii) we have

$$\psi = AEHJ - AFGI = 2AEHJ - AFGI + AF^2J - AF^2J =$$

$$AJ(2EH + F^2) - AF(GI + FJ) \in (V)_4.$$

This completes the proof of the equation (4) for $n = 4$. When n is larger than 4 then by the induction assumption we can assume that the equation (3) holds for all integers $2 \leq k \leq n - 1$. Again we use the Remark 1.3. Let $\beta = \beta_1 + \beta_2 \in \text{Ann}(\det(X)) \cap S_4^s$. If the two terms, β_1 and β_2 have a common factor l , i.e. $\beta_1 = la_1$ and $\beta_2 = la_2$, then $\beta = l(a_1 + a_2)$ where a_1 and a_2 are of degree at most $n - 1$. Now by the induction assumption the proposition holds for the binomial $a_1 + a_2$, i.e. $a_1 + a_2 \in V_{n-1}$ hence we have

$$\beta = l(a_1 + a_2) \in l(V)_{n-1} \subset (V)_n.$$

If the two terms, β_1 and β_2 do not have any common factor then with the same method as used in (iii), we can rewrite the binomial β by adding and subtracting a term m of degree n , which has a common factor m_1 with β_1 and a common factor m_2 with β_2 , now we will have

$$\beta_1 + \beta_2 = \beta_1 + m + \beta_2 - m = m_1(c_1 + m') + m_2(c_2 - m''),$$

where $\beta_1 = m_1c_1$, $m = m_1m' = m_2m''$ and $\beta_2 = m_2c_2$. Now $c_1 + m'$ and $c_2 - m''$ are of degree at most $n - 1$ so by the induction assumption we have

$$\beta_1 + \beta_2 = m_1(c_1 + m') + m_2(c_2 - m'') \in (V)_n.$$

This completes the induction step and the proof of the proposition. □

Recall that for the generic symmetric $n \times n$ matrix X , the unacceptable monomials of degree k in S_k^s are monomials which do not divide any term of the determinant of X . We denote the set of degree k unacceptable monomials by U_k .

Lemma 3.7. *We can write each unacceptable monomial of degree k ($2 \leq k \leq n$), as an explicit element of $S_{k-2}^s V$, where V is the space defined in the Proposition 3.1.*

Proof. We use induction on k . For $k = 2$ the claim is obviously true. We show that the claim is true for $k = 3$. We need to show that the space U_3 of unacceptable monomials in S_3^s are in $S_1^s V$. The unacceptable monomials of degree 3 for the $n \times n$ generic symmetric matrix have one of the following forms:

(a) The form x^2y where x is a diagonal element.

(b) The form xyz where x is a diagonal element, $y \neq x$ is in the same row or column with x and $z \neq x$.

(c) The form xyz where x, y, z are non diagonal elements from the same row or column (can be equal to each other).

Unacceptable monomials of type (a) or (b) are multiples of unacceptable monomials of degree 2, so they are in the space $S_1^s U_2$. So we only need to show that the degree 3 unacceptable monomials of type (c) are in $S_1^s V$. The 3 non diagonal elements in the same row or column of the matrix X are from a symmetric 4×4 sub-matrix. So without loss of generality we show that a degree 3 monomial of type (c) from the following sub-matrix is in $S_1^s V$. Let A^s be the 4×4 symmetric sub-matrix of a generic symmetric $n \times n$ matrix,

$$A^s = \begin{pmatrix} a & b & c & d \\ b & e & f & g \\ c & f & h & i \\ d & g & i & j \end{pmatrix},$$

and D^s be the matrix

$$D^s = \begin{pmatrix} A & B & C & D \\ B & E & F & G \\ C & F & H & I \\ D & G & I & J \end{pmatrix}$$

Monomials of type (c) in S_3^s annihilating $\det(A^s)$ can be of one of the following forms:

(I) All three non-diagonal variables are distinct. Consider the monomial $\eta_1 = BCD$, a degree three unacceptable monomial of type (c). We have $AF + BC \in V$ (since it is a permanent with one diagonal element), so we have $(AF + BC) \circ \det(A^s) = 0$. Hence,

$$D(AF + BC) \circ \det(A^s) = 0.$$

We also know that $DAF \in S_1^s U_2 \subset U_3$ so $DAF \circ \det(A^s) = 0$. We have $\eta_1 = BCD = D(AF + BC)(\text{mod } S_1^s U_2)$. So we have $\eta_1 \in S_1^s V$.

(II) There are only two distinct non-diagonal variables. Consider the monomial $\eta_2 = B^2 C$, also of type (c). We have $AF + BC \in V$ (since it is a permanent with one diagonal element), so we have $(AF + BC) \circ \det(A^s) = 0$. Hence,

$$B(AF + BC) \circ \det(A^s) = 0.$$

We also know that $BAF \in S_1^s U_2 \subset U_3$ so $B^2 C \circ \det(A^s) = 0$. We have $\eta_2 = B^2 C = B(AF + BC)(\text{mod } S_1^s U_2)$. So we have $\eta_2 \in S_1^s V$.

(III) There is only one non-diagonal variable. Consider the monomial $\eta_3 = B^3$, also of type (c). We have $B^2 + 2AE \in V$ (since it is a diagonal permanent with the coefficient 2), so we have $(B^2 + 2AE) \circ \det(A^s) = 0$. Hence,

$$B(B^2 + 2AE) \circ \det(A^s) = 0.$$

We also know that $BAE \in S_1^s U_2 \subset U_3$ so $B^3 \circ \det(A^s) = 0$. We have $\eta_3 = B^3 = B(B^2 + 2AE)(\text{mod } S_1^s U_2)$. So we have $\eta_3 \in S_1^s V$.

So the lemma is true for $k = 3$ and we have $U_3 \subset S_1^s V$. Let P denote the subspace of V generated by binomials of type (b) and (c) defined in Proposition 3.1. We have shown that $U_3 \subset S_1^s(U + P)$.

Now assume that $k \geq 4$ and the lemma is true for all integers less than k . We want to show that the claim is true for k . Let $\mu = \mu_1 \mu_2 \dots \mu_k$ be an unacceptable monomial of degree k . We can write μ such that $\mu_2 \dots \mu_k$ is an unacceptable monomial of degree $k - 1$ so we have

$$\mu = \mu_1(\mu_2 \dots \mu_k) \in S_1^s(S_{k-3}^s V) = S_{k-2}^s V.$$

So the lemma is true also for k . □

Notation. We use the following definitions and notations in the remaining part of this section.

- By Lexicographic/Conca order we mean the lexicographic term order induced by the variable order,

$$Y_{1,1} > Y_{1,2} > \dots > Y_{1,n} > Y_{2,2} > \dots > Y_{2,n} > \dots > Y_{n-1,n} > Y_{n,n}.$$

By Reverse Lexicographic order we mean the lexicographic term order induced by the variable order,

$$Y_{1,1} < Y_{1,2} < \dots < Y_{1,n} < Y_{2,2} < \dots < Y_{2,n} < \dots < Y_{n-1,n} < Y_{n,n}.$$

- Let \mathbf{M} be the $k \times k$ minor of the generic symmetric matrix X , with the set of rows $\{a_1, \dots, a_k\}$ and the set of the columns $\{b_1, \dots, b_k\}$ where $a_1 < \dots < a_k$ and $b_1 < \dots < b_k$. Then the initial monomial of \mathbf{M} using the lexicographic (Conca) order is $x_{a_1 b_1} \dots x_{a_k b_k}$.
- We denote by $[a_1, \dots, a_k | b_1, \dots, b_k]$ the $k \times k$ doset minor with the sequence of rows $a = (a_1, \dots, a_k)$ and the sequence of columns $b = (b_1, \dots, b_k)$ both subsequences of $(1, \dots, n)$ satisfying the following conditions:

$$a_1 < a_2 < \dots < a_k,$$

$$b_1 < b_2 < \dots < b_k,$$

$$a_i \leq b_i, \forall 1 \leq i \leq k.$$

- We denote by $(a_1, \dots, a_k | b_1, \dots, b_k)$ the acceptable monomial $x_{a_1 b_1} \dots x_{a_k b_k}$. Note that we write the acceptable monomial $m = (a_1, \dots, a_k | b_1, \dots, b_k)$, with $a = (a_i)$ an increasing sequence. But unlike the doset minors, the sequence $b = (b_i)$ doesn't need to be increasing.
- For the monomial $m = x_{a_1 b_1} \dots x_{a_k b_k}$ denoted by $(a_1, \dots, a_k | b_1, \dots, b_k)$, we call a pair (b_i, b_j) , with $i < j$, a reversal pair if $b_i \geq b_j$.
- Let $\mu = [i_1, \dots, i_k | j_1, \dots, j_k]$ be a $k \times k$ doset minor. We call the monomial $m_f = x_{i_1 j_1} \dots x_{i_k j_k}$ the flag monomial of μ .
- “Conca monomial” is the initial monomial of a doset minor in lexicographic order. Note that the initial monomial of a doset minor in lexicographic order is the flag monomial of that minor.
- The set of all $k \times k$ doset minors form a Gröbner basis for the ideal generated by all $k \times k$ minors (Theorem 2.2). Hence the ideal generated by the set of initial monomials of all minors is equal to the ideal generated by the set of the initial monomials of the doset minors.
- Let A_k be the set of acceptable monomials in S_k^s .
- Let $\iota : R^s \rightarrow S^s$, $\iota(x_{ij}) = y_{ij}$, and C_k be the subset of A_k defined by

$\{\iota(\mu) | \mu \text{ a Conca initial monomial (in Lex. order) of a } k \times k \text{ doset minor of } X\}$.

- Let C'_k be the complementary set to C_k of acceptable monomials in A_k .
- For each $\mu \in A_k$, let $A_{>\mu}$, denote the subset of elements $\nu \in A_k$, such that $\nu > \mu$ in the lexicographic order of S^s .

Proposition 3.8. *Each acceptable non-Conca monomial of degree k ($3 \leq k \leq n$), is the initial monomial (in the reverse Lex. order) of an element of $S_{k-2}^s V$.*

Proof. We use the induction on $k \geq 3$. First let $k = 3$. Let μ be a degree 3 acceptable monomial which is not the initial term of any 3×3 doset minor in the lexicographic order. The acceptable monomials, $y_{i_1 i_2} y_{i_3 i_4} y_{i_5 i_6}$, of degree 3 for the $n \times n$ generic symmetric matrices can be listed as follows:

- (a) All 6 indices are distinct
- (b) There is one repeated index
- (c) There are 2 repeated indices
- (d) There are 3 repeated indices

Now we discuss each of the above types separately.

(a) all 6 indices are distinct $m = y_{i_1 i_2} y_{i_3 i_4} y_{i_5 i_6}$, $(i_1, i_3, i_5 | i_2, i_4, i_6)$. Without loss of generality we can assume these indices are 1,2,3,4,5,6. In order to have a non-Conca monomial of this kind, it is enough to have at least one reversal pair. Since the monomial m has a reversal pair it is not the initial monomial of any 3×3 minor of Y . So it is not in the ideal generated by all the initial monomials of the 3×3 minors of Y . Hence by Theorem 2.2 it is not in the ideal generated by all the initial monomials of all 3×3 doset minors of Y . For example in $(1, 2, 3 | 6, 4, 5)$, $6 \geq 4$ so $(6, 4)$ is a reversal pair.

In the doset minor $[i_1, i_3, i_5 | i_2, i_4, i_6]$, we have

$$i_1 < i_3 < i_5,$$

$$i_2 < i_4 < i_6.$$

Now assume $i_1 = 1, i_3 = 2, i_5 = 3, i_2 = 4, i_4 = 5$ and $i_6 = 6$ then $y_{14}y_{25}y_{36}$ is the initial term in the corresponding 3×3 doset minor using the lexicographic order. Now we consider a non-initial Conca monomial which has at least one reversal pair $(i_j, i_k), (j < k)$ where $j, k \in \{2, 4, 6\}$ such that $i_j \geq i_k$.

Now we look at the corresponding 6×6 symmetric sub-matrix. we have

$$X = \begin{pmatrix} a & b & c & d & e & f \\ b & g & h & i & j & k \\ c & h & l & m & n & o \\ d & i & m & p & q & r \\ e & j & n & q & s & t \\ f & k & o & r & t & u \end{pmatrix},$$

$$Y = \begin{pmatrix} A & B & C & D & E & F \\ B & G & H & I & J & K \\ C & H & L & M & N & O \\ D & I & M & P & Q & R \\ E & J & N & Q & S & T \\ F & K & O & R & T & U \end{pmatrix},$$

Consider a non-Conca degree three monomial involving 6 distinct rows and columns. Without loss of generality we consider the monomial $(1, 2, 5 | 6, 4, 3)$. We have the Hafnian of the following 4×4 symmetric sub-matrix with the rows and columns 1,2,4,6,

$$\text{Haf} \begin{pmatrix} A & B & D & F \\ B & G & I & K \\ D & I & P & R \\ F & K & R & U \end{pmatrix} = BR + DK + FI,$$

Given $\mu = FIN$, $f_\mu = N(BR + DK + FI) \in S_1^s V$, where $N(BR + DK) \in A_{>\mu}$.

(b) There is one repeated index. Without loss of generality we can assume these indexes are 1,2,3,4,5, with one of them repeated. In order to have a non-Conca example of this kind, it is enough to have 1 reversal pair. For example in $(1, 2, 3|4, 1, 5)$, $4 > 1$. We can form a 5×5 symmetric matrix with these rows and columns,

$$X = \begin{pmatrix} a & b & c & d & e \\ b & f & g & h & i \\ c & g & j & k & l \\ d & h & k & m & n \\ e & i & l & n & o \end{pmatrix},$$

$$Y = \begin{pmatrix} A & B & C & D & E \\ B & F & G & H & I \\ C & G & J & K & L \\ D & H & K & M & N \\ E & I & L & N & O \end{pmatrix},$$

Now we consider the monomial $\mu = y_{14}y_{21}y_{35} = BDL$ as an acceptable monomial of type (b). We have the Hafnian of the following 4×4 symmetric sub-matrix with the rows and columns 1,2,3,5,

$$\text{Haf} \begin{pmatrix} A & B & C & E \\ B & F & G & I \\ C & G & J & L \\ E & I & L & O \end{pmatrix} = BL + CI + EG,$$

Given $\mu = BDL$, $f_\mu = D(BL + CI + EG) \in S_1^s V$, where $D(CI + EG) \in A_{>\mu}$.

(c) There are 2 repeated indices, Without loss of generality we can assume these indexes are 1,2,3,4, with two of them repeated. In order to have a non-Conca example of this kind, it is enough to have one reversal pair. For example in $(1, 2, 3|2, 1, 4)$, $2 > 1$. We can form a 4×4 symmetric matrix with these rows and columns,

$$X = \begin{pmatrix} a & b & c & d \\ b & e & f & g \\ c & f & h & i \\ d & g & i & j \end{pmatrix},$$

$$Y = \begin{pmatrix} A & B & C & D \\ B & E & F & G \\ C & F & H & I \\ D & G & I & J \end{pmatrix},$$

Now we consider the monomial $\mu = y_{12}y_{21}y_{34} = B^2I$ as an acceptable monomial of type (c). Given $\mu = B^2I$, $f_\mu = I(B^2 + 2AE) \in S_1^s V$, where $AEI \in A_{>\mu}$.

(d) There are 3 repeated indices, Without loss of generality we can assume these indexes are 1,2,3, all of them repeated. In order to have a non-Conca example of this kind, it is enough to have 1 reversal pair. For example in $(1, 2, 3|3, 2, 1)$, $3 > 1$. We can form a 3×3 symmetric matrix with these rows and columns,

$$X = \begin{pmatrix} a & b & c \\ b & d & e \\ c & e & f \end{pmatrix},$$

$$Y = \begin{pmatrix} A & B & C \\ B & D & E \\ C & E & F \end{pmatrix},$$

Now we consider the monomial $\mu = y_{13}y_{22}y_{31} = C^2D$ as an acceptable monomial of type (c). Given $\mu = C^2D$, $f_\mu = C(BE + CD) \in S_1^s V$, where $BEC \in A_{>\mu}$.

Since other cases are similar to the above examples, for $k = 3$ the claim of Proposition 3.8 is true. Now assume that the Proposition 3.8 is true for all integers less than k . We have to show that the Proposition 3.8 is also true for k . Let $\mu = y_{i_1 j_1} \dots y_{i_k j_k} = (i_1, \dots, i_k | j_1, \dots, j_k)$ be a degree k acceptable non-Conca monomial, so it has at least one reversal pair. Now we can consider μ as the product of one variable, y_{ab} and a degree $k - 1$ acceptable non-Conca monomial, μ_1 , containing at least one reversal pair. Then by the induction assumption μ_1 is the initial monomial (in rev. lex.) of an element of $S_{k-3}^s V$. So we have $\mu = y_{ab}\mu_1$ as the initial monomial (in rev. lex.) of an element of $S_{k-2}^s V$. This completes the proof. \square

Example 3.9. Consider the case $n = 3$. we have

$$X = \begin{pmatrix} a & b & c \\ b & d & e \\ c & e & f \end{pmatrix},$$

$$Y = \begin{pmatrix} A & B & C \\ B & D & E \\ C & E & F \end{pmatrix},$$

There are 5 acceptable degree 3 monomials. Using the lexicographic term order induced by the variable order,

$$Y_{1,1} > Y_{1,2} > Y_{1,3} > Y_{2,2} > Y_{2,3} > Y_{3,3},$$

we have the following order on the degree 3 acceptable monomials

$$ADF > AE^2 > B^2F > BEC > C^2D.$$

- The set of Conca initial monomials of degree three, C_3 , is the subspace spanned by the set $\{ADF\}$.
- The set of all acceptable degree three monomials that are not in C_3 is

$$C'_3 = \{AE^2, B^2F, BEC, C^2D\}.$$

- For $\mu_1 = C^2D$, $f_{\mu_1} = C^2D + 2ADF = D(C^2 + 2AF) \in S_1^s V$, where $ADF \in A_{>\mu_1}$.
- For $\mu_2 = BCE$, $f_{\mu_2} = BEC + AE^2 = E(BC + AE) \in S_1^s V$, where $AE^2 \in A_{>\mu_2}$.
- For $\mu_3 = B^2F$, $f_{\mu_3} = B^2F + 2ADF = F(B^2 + 2AD) \in S_1^s V$, where $ADF \in A_{>\mu_3}$.
- For $\mu_4 = AE^2$, $f_{\mu_4} = A(E^2 + 2DF) \in S_1^s V$, where $ADF \in A_{>\mu_4}$.

Hence each acceptable non-Conca monomial of degree three is the initial monomial (in the reverse Lex. order) of an element of $S_1^s V$.

Corollary 3.10. For $1 \leq k \leq n$ we have

$$(V)_k = \text{Ann}(\det(X)) \cap S_k^s.$$

We also have $(V)_{n+1} = S_{n+1}^s$.

Proof. By equation (4) we have

$$S_{k-2}^s V = (V)_k \subset \text{Ann}(\det(X)) \cap S_k^s.$$

By Remark 1.2 and Lemma 3.2 we have

$$(\text{Ann}(\det(X)))_k = (\text{Ann}(S_{n-k}^s \circ (\det(X))))_k = (\text{Ann}(M_k(X)))_k$$

So we have

$$\dim S_{k-2}^s V \leq \dim(\text{Ann}(\det(X)) \cap S_k^s) = \dim S_k^s - \dim M_k(X).$$

On the other hand, by definition the sets U_k and C'_k are linearly independent and form a basis for the corresponding subspaces. Hence by Lemma 3.7 and Proposition 3.8 we have

$$\dim S_k^s - \dim M_k(X) = \dim \langle C'_k \rangle + \dim \langle U_k \rangle \leq \dim S_{k-2}^s V.$$

So we have

$$\dim(V)_k = \dim S_{k-2}^s V = \dim S_k^s - \dim M_k(X) = \dim(\text{Ann}(\det(X)) \cap S_k^s).$$

□

Theorem 3.11. *Let X be a generic symmetric $n \times n$ matrix. Then the apolar ideal $\text{Ann}(\det(X))$ is the ideal (V) generated in degree 2.*

Proof. This follows directly from Lemma 3.7, Proposition 3.8 and Corollary 3.10. □

Proposition 3.12. *The set V is a Gröbner basis for the ideal $\text{Ann}(\det(X))$.*

Proof. We have shown that V generates $\text{Ann}(\det(X))$. Now we use Buchberger's Algorithm to show that V is a Gröbner basis for the ideal $\text{Ann}(\det(X))$.

(1) Let \mathcal{F} and \mathcal{G} and be two distinct permanents of Y of type (c) in Proposition 3.1. Let $\mathcal{F} = y_{ii}y_{jk} + y_{ik}y_{ji}$ and $\mathcal{G} = y_{uu}y_{zv} + y_{uv}y_{zu}$.

$$\mathcal{F} = \text{perm} \begin{pmatrix} y_{ii} & y_{ik} \\ y_{ji} & y_{jk} \end{pmatrix}.$$

$$\mathcal{G} = \text{perm} \begin{pmatrix} y_{uu} & y_{uv} \\ y_{zu} & y_{zv} \end{pmatrix}.$$

Let $f_1 = y_{ii}y_{jk}$ be the leading term of \mathcal{F} , and $g_1 = y_{uu}y_{zv}$ be the leading term of \mathcal{G} with respect to Conca monomial order. Denote the least common multiple of f_1 and g_1 by h . Then we have:

$$S(\mathcal{F}, \mathcal{G}) = (h/f_1)F - (h/g_1)G = y_{uu}y_{zv}y_{ik}y_{ji} - y_{ii}y_{jk}y_{uv}y_{zu}.$$

Now using the multivariate division algorithm, we reduce $S(\mathcal{F}, \mathcal{G})$ relative to the set V . When there is no common factor in the initial terms of \mathcal{F} and \mathcal{G} the reduction is zero. First we reduce $S(\mathcal{F}, \mathcal{G})$ dividing by \mathcal{F} , so we will have

$$S(\mathcal{F}, \mathcal{G}) + y_{uv}y_{zu}\mathcal{F} = y_{uu}y_{zv}y_{ik}y_{ji} + y_{uv}y_{zu}y_{ik}y_{ji}.$$

Then we reduce the result using \mathcal{G} this time, so we will have

$$y_{uu}y_{zv}y_{ik}y_{ji} + y_{uv}y_{zu}y_{ik}y_{ji} - y_{ik}y_{ji}\mathcal{G} = 0.$$

So we have shown that for all pairs \mathcal{F} and \mathcal{G} of distinct permanents of Y of type (c), the S -polynomials $S(\mathcal{F}, \mathcal{G})$ reduce to zero with respect to V .

(2) Let $\mathcal{F} = y_{ii}y_{jk} + y_{ik}y_{ji}$ and $\mathcal{G} = y_{ii}y_{lm} + y_{im}y_{li}$ be two permanents whose initial terms have a common factor. We have

$$S(\mathcal{F}, \mathcal{G}) = y_{lm}y_{ik}y_{ji} - y_{jk}y_{im}y_{li}.$$

Without loss of generality we can restrict to a 5×5 symmetric sub-matrix. Note that in a 5×5 symmetric sub-matrix we can have two Hafnians whose initial term has one common factor, two permanents whose initial terms have a common factor, and a permanent and a Hafnian whose initial terms have a common factor.

$$\begin{pmatrix} A & B & C & D & E \\ B & F & G & H & I \\ C & G & J & K & L \\ D & H & K & M & N \\ E & I & L & N & O \end{pmatrix}.$$

Without loss of generality we consider the two permanents $\mathcal{F} = AG + BC$ and $\mathcal{G} = AN + DE$.

$$S(\mathcal{F}, \mathcal{G}) = BCN - DEG.$$

We checked using the multivariate division algorithm in Macaulay 2 that the binomial $BCN - DEG$ reduces to zero mod the initial set of generators V .

Note that any two 4×4 Hafnians with the same initial term are exactly the same. So for the Hafnians it is enough to consider the S -polynomials of Hafnians whose initial terms have only one common factor, and of the Hafnians whose initial terms do not have a common factor. We should also consider the S -polynomials in the case that we have a Hafnian and a permanent.

(3) Let \mathcal{F} and \mathcal{G} and be two distinct Hafnians of Y whose initial terms do not have a common factor. Without loss of generality we can restrict to a 5×5 symmetric matrix as in (2),

Without loss of generality we consider the two Hafnians $\mathcal{F} = HL + IK + GN$ and $\mathcal{G} = DG + CH + BK$.

$$S(\mathcal{F}, \mathcal{G}) = BKHL + BIK^2 - DG^2N - CGHN.$$

The multivariate division algorithm in Macaulay 2 shows that the S -polynomial, $BKHL + BIK^2 - DG^2N - CGHN$, reduces to zero.

(4) Let \mathcal{F} and \mathcal{G} and be two distinct Hafnians of Y whose initial terms have a common factor. Without loss of generality we can restrict to a 5×5 symmetric matrix as in (2),

Without loss of generality we consider the two Hafnians $\mathcal{F} = CH + DG + BK$ and $\mathcal{G} = CI + EG + BL$.

$$S(\mathcal{F}, \mathcal{G}) = CHL + DGL - CIK - EGK.$$

Using the multivariate division algorithm in Macaulay 2, it is easy to see that the polynomial, $CHL + DGL - CIK - EGK$, reduces to zero. We also show the reduction process for this example directly, We want to reduce the polynomial $S(\mathcal{F}, \mathcal{G})$ using the set V . The initial term for this polynomial is CHL . So we should find elements of the set V other than \mathcal{F} and \mathcal{G} whose initial terms divide CHL . We have the following three possibilities:

- (a) The initial term is CH . There is no permanent or Hafnian with this initial term in the set V .
- (b) The initial term is CL . The only element of the set V with this initial term is the permanent $CL + EJ$
- (c) The initial term is HL . There is no permanent or Hafnian with this initial term in the set V .

So we reduce $S(\mathcal{F}, \mathcal{G})$ using $CL + EJ$, and we get

$$S' = S(\mathcal{F}, \mathcal{G}) - H(CL + EJ) = -CIK + DGL - EGK - JHE.$$

Now the initial term of S' is CIK , and we again do the reduction process. Here we have three different possibilities to choose an element from V .

(a') The initial term is CI . There is no permanent or Hafnian with this initial term in the set V .

(b') The initial term is CK . The only element of the set V with this initial term is the permanent $CK + DJ$

(c') The initial term is IK . There is no permanent or Hafnian with this initial term in the set V .

So we reduce S' using $CK + DJ$, and we get

$$S'' = S' - I(CK + DJ) = DGL - EGK - JHE + DIJ.$$

Again we look at the three different degree 2 monomials which divide the initial term of S'' , we have

(a'') The initial term is DG . There is no permanent or Hafnian with this initial term in the set V .

(b'') The initial term is GL . The only element of the set V with this initial term is the permanent $GL + IJ$

(c'') The initial term is DL . There is no permanent or Hafnian with this initial term in the set V .

So we reduce S'' using $GL + IJ$, and we get

$$S''' = S'' - D(GL + IJ) = -E(GK + HJ) \in V.$$

So the S -polynomial can be reduced to zero using the set V .

(5) Let \mathcal{F} be a permanent and \mathcal{G} be a Hafnian of Y whose initial terms do not have a common factor. Without loss of generality we can restrict to a 5×5 symmetric matrix as in (2),

Without loss of generality we consider two permanents $\mathcal{F} = 2FJ + G^2$ and $\mathcal{G} = CI + EG + BL$.

$$S(\mathcal{F}, \mathcal{G}) = BLG^2 - 2FJCI - 2FJEG.$$

The multivariate division algorithm in Macaulay 2 shows that the S -polynomial, $BLG^2 - 2FJCI - 2FJEG$ can be reduced to zero using the set V .

(6) Let \mathcal{F} be a permanent and \mathcal{G} be a Hafnian of Y whose initial terms have a common factor. Without loss of generality we can restrict to a 5×5 symmetric matrix as in (2),

Without loss of generality we consider two permanents $\mathcal{F} = BG + CF$ and $\mathcal{G} = CI + EG + BL$.

$$S(\mathcal{F}, \mathcal{G}) = CFL - CIG - EG^2.$$

The multivariate division algorithm in Macaulay 2 shows that the S -polynomial, $CFL - CIG - EG^2$ reduces to zero.

□

3.2 Apolar ideal of the permanent

In this section we determine the apolar ideal of the permanent of the $n \times n$ generic symmetric matrix, and we will show that it is generated by degree two and degree three polynomials. We first determine the generators of degree two (Proposition 3.13). We then determine the degree three generators, which occur when $n \geq 6$ (Lemma 3.18). A key step is to use triangularity to show that these degree two and degree three generators, generate all the apolar ideal (Lemma 3.19 and Proposition 3.21). This leads to our main result (Theorem 3.23).

Analogous to Proposition 3.1 we have:

Proposition 3.13. *For an $n \times n$ symmetric matrix $X = (x_{ij})$, $\text{Ann}(\text{Perm}(X)) \subset S^s$, includes the following degree 2 polynomials:*

- (a) *Unacceptable monomials including $y_{ii}y_{ij}$ for all $1 \leq i, j \leq n$. The number of these monomials are n^2 .*
- (b) *All the diagonal 2×2 minors with a coefficient 2 on the diagonal term, i.e. $y_{ij}^2 - 2y_{ii}y_{jj}$. The number of these binomials is $\binom{n}{2}$.*

(c) All the 2×2 minors with one diagonal element, i.e. $y_{jk}y_{il} - y_{jl}y_{ik}$. The number of these binomials is $n \cdot \binom{n-1}{2}$.

Proof. We have $\text{Perm}(X) = \sum_{\sigma \in S_n^s} \prod x_{i,\sigma(i)}$. First we show that monomials of type (a) are in $\text{Ann}(\text{Perm}(X))$. By symmetry we have

$$\begin{aligned} y_{ii}y_{ij} \circ \text{Perm}(X) &= 0 \text{ (where } j \geq i), \\ y_{ii}y_{ji} \circ \text{perm}(X) &= 0 \text{ (where } j \leq i). \end{aligned}$$

Now we want to show that the binomials of type (b) are in $\text{Ann}(\text{perm}(X))$. Let $M = 2y_{ii}y_{jj} - y_{ij}^2$.

There are $n!$ terms in the expansion of the permanent. If a term, t , does not contain the monomial $x_{ii}x_{jj}$ or the monomial x_{ij}^2 then $M \circ t = 0$. Let σ_1 be a permutation having x_{ii} and x_{jj} respectively in its i -th and j -th place. Corresponding to σ_1 we also have a permutation $\sigma_2 = \tau\sigma_1$, where $\tau = (i, j)$ is a transposition. Thus the resulting action of the minor M on $\text{perm}(X)$ is zero.

To show that the binomials of type (c) are in the annihilator ideal we can use a similar proof so that we used for the binomials of type (b).

□

We denote by $\{W\}$ be set of the degree 2 elements of type (a), (b) and (c) in Proposition 3.13, and by W the vector subspace of S^s spanned by $\{W\}$. We denote by $\{a\}$, $\{b\}$, and $\{c\}$ the set of elements in (a), (b), and (c) respectively.

Analogous to Lemma 3.2 we have:

Lemma 3.14. *The set $\{W\}$ is linearly independent and we have,*

$$\dim_k W = n^2 + \binom{n}{2} + n \cdot \binom{n-1}{2}.$$

Proof. Each of the three subsets are linearly independent from each other since they involve different variables. So if we show that each subset is linearly independent we are done. The subset $\{a\}$ is linearly independent since the monomials in $\{a\}$ form a Gröbner basis for the ideal they generate. The subsets $\{b\}$ and $\{c\}$ are linearly independent since by choosing two elements of the matrix, where at least one element is diagonal, we have a unique 2×2

minor. Hence the set $\{W\}$ is linearly independent and the dimension of the vector space W is $n^2 + \binom{n}{2} + n \cdot \binom{n-1}{2}$.

□

Notation. For a generic symmetric $n \times n$ matrix X , we denote by $P_k(X)$ the space of the permanents of all $k \times k$ sumatrices of X .

Analogous to Lemma 3.3 we have:

Lemma 3.15. *Let $1 \leq k \leq n$. We have*

$$S_k^s \circ (\text{perm}(X)) = P_{n-k}(X) \subset R^s.$$

Proof. To show the inclusion

$$S_k^s \circ (\text{perm}(X)) \subset P_{n-k}(X) \subset R^s,$$

we use induction on k . Let P_{ij} denote the permanent of the sub matrix obtained by deleting the i -th row and j -th column. For $k = 1$, we have two different cases:

I) for a diagonal element y_{ii} we have

$$y_{ii} \circ (\text{perm}(X)) = P_{ii} \in P_{n-1}(X).$$

II) Let y_{ij} be a non-diagonal element. Without loss of generality we can consider y_{12} . We have $y_{12} = y_{21}$. The monomial y_{12}^2 appears in exactly $(n-2)!$ terms coming from $y_{12}^2 \cdot (P_{12})_{21}$.

We also have $2((n-1)! - (n-2)!)$ terms in the permanent which contain y_{12} but do not contain y_{12}^2 . These terms come from the sub-permanent obtained by deleting one of the rows 1 or 2.

So we have

$$y_{ij} \circ (\text{perm}(X)) = 2P_{ij} \in P_{n-1}(X).$$

Now assume that the above inclusion holds for $k-1$, i.e

$$S_{k-1}^s \circ (\text{perm}(X)) \subset P_{n-(k-1)}(X),$$

and we want to show that it is true for k . We have

$$S_k^s \circ \text{perm}(X) = S_1^s S_{k-1}^s \circ \text{perm}(X) \subset S_1^s \circ P_{n-k+1}(X) \subset P_{n-k}(X).$$

Now we want to show the opposite inclusion

$$S_k^s \circ (\text{perm}(X)) \supset P_{n-k}(X) \subset R^s,$$

Let $P_{\widehat{I}, \widehat{J}}(X)$, $I = \{i_1, \dots, i_k\}$, $J = \{j_1, \dots, j_k\}$, $1 \leq i_1 \leq i_2 \leq \dots \leq i_k \leq n$, $1 \leq j_1 \leq j_2 \leq \dots \leq j_k \leq n$ be the $(n-k) \times (n-k)$ permanent of the submatrix of X one obtains by deleting the I rows and J columns of X . Let

$$\Delta_{(I,J)} = \{(i_r, j_r) | i_r \in I, j_r \in J \text{ and } i_r = j_r\}.$$

Let $\Delta_I = \{i_r | (i_r, j_r) \in \Delta_{(I,J)}\}$ and $\Delta_J = \{j_r | (i_r, j_r) \in \Delta_{(I,J)}\}$. Let $P_{(I,J)-\Delta}$ be the submatrix of Y with the rows $I - \Delta_I$, and the columns $J - \Delta_J$.

Claim:

$$P_{\widehat{I}, \widehat{J}} = c \prod_{(i_r, j_r) \in \Delta_{(I,J)}} y_{i_r j_r} \text{perm}(P_{(I,J)-\Delta}) \circ (\text{perm}(X))$$

where $c \neq 0 \in \mathbf{k}$.

To prove the claim we use induction on $|I| = |J| = k$, the cardinality of the sets I and J . First we show the claim is true for $k = 1$. Let $I = \{i_1\}$ and $J = \{j_1\}$. We have two cases

I. $i_1 = j_1$ so $y_{i_1 j_1}$ is a diagonal element and we have

$$P_{\widehat{I}, \widehat{J}} = y_{i_1 j_1} \circ (\text{perm}(X)).$$

II. $i_1 \neq j_1$ so we have

$$y_{i_1 j_1} \circ (\text{perm}(X)) = 2P_{\widehat{I}, \widehat{J}}.$$

So for $k = 1$ the claim holds. Now assume that the claim holds for every I and J with $|I| = |J| = k - 1$ and we want to show that the claim is also true for I and J with $|I| = |J| = k$.

Let $I = \{i_1, \dots, i_k\}$ and $J = \{j_1, \dots, j_k\}$.

Let $I' = I - \{i_1\}$ and $J' = J - \{j_1\}$. We have $|I'| = |J'| = k - 1$ so by the induction assumption we have

$$P_{\widehat{I'}, \widehat{J'}} = c \prod_{(i_r, j_r) \in \Delta_{(I', J')}} y_{i_r j_r} \text{perm}(P_{(I', J')-\Delta}) \circ (\text{perm}(X))$$

Now writing the Laplace expansion of the permanent using row i_1 or column j_1 for $P_{\hat{I},\hat{J}}$, we get

$$P_{\hat{I},\hat{J}} = c \prod_{(i_r,j_r) \in \Delta_{(I,J)}} y_{i_r j_r} \text{perm}(P_{(I,J)-\Delta}) \circ (\text{perm}(X)),$$

where $c \neq 0 \in \mathbb{k}$. Hence $P_{\hat{I},\hat{J}} \in S_{n-k}^s \circ (\text{perm}(X))$.

□

Lemma 3.16. *Denote by P_2 the space of 2×2 permanents of the $n \times n$ generic symmetric matrix X . We have*

$$\dim_{\mathbb{k}} P_2 = \frac{\binom{n}{2}(\binom{n}{2} + 1)}{2}.$$

Proof. By lemma 3.2 the dimension of the subspace of 2×2 permanents with at least one diagonal element is

$$\binom{n}{2} + n \cdot \binom{n-1}{2}.$$

Now the number of distinct 2×2 permanents with no diagonal element is

$$\frac{\binom{n}{4}\binom{4}{2}}{2} = 3 \cdot \binom{n}{4}$$

Each of the above subsets are linearly independent from each other since they involve different variables. We have shown in Lemma 3.2 that the subspace of 2×2 permanents with at least one diagonal element is linearly independent. The subspace of 2×2 permanent with no diagonal element is also linearly independent, since by choosing two different rows, i and j and then choosing two different columns l and k such that $i \neq j \neq l \neq k$ we have a unique 2×2 permanent. Hence we have

$$\dim_{\mathbb{k}} P_2 = \binom{n}{2} + n \cdot \binom{n-1}{2} + 3 \cdot \binom{n}{4} = \frac{\binom{n}{2}(\binom{n}{2} + 1)}{2}.$$

□

Analogous to Lemma 3.4 we have

Lemma 3.17. *For the generic symmetric $n \times n$ matrix X , we have*

$$W = \text{Ann}(P_2) \cap S_2^s = (\text{Ann}(\text{perm}(X)))_2$$

Proof. By the Lemma 3.15 we have

$$\text{Ann}(S_{n-2}^s \circ (\text{perm}(X))) = \text{Ann}(P_2(X)).$$

By the Proposition 3.13 we have

$$(\text{Ann}(\text{perm}(X)))_2 \supset W$$

By the Remark 1.2 we have

$$(\text{Ann}(\text{perm}(X)))_2 = (\text{Ann}(S_{n-2}^s \circ (\text{perm}(X))))_2 \subset \text{Ann}(P_2(X))$$

Hence we have

$$W \subset \text{Ann}(P_2).$$

On the other hand, using Lemma 3.16 we have

$$\dim(W) = n^2 + \binom{n}{2} + n \cdot \binom{n-1}{2} = \binom{\frac{n^2+n}{2} + 1}{2} - \frac{\binom{n}{2}(\binom{n}{2} + 1)}{2} = \dim S_2^s - \dim P_2(X).$$

So we have

$$W = \text{Ann}(P_2) \cap S_2^s.$$

By the Proposition 3.13, Lemmas 3.14 and 3.15 we have

$$(\text{Ann}(\text{perm}(X)))_2 \subset \text{Ann}(P_2(X)) \subset W.$$

So we have

$$W = (\text{Ann}(\text{perm}(X)))_2.$$

□

The apolar ideal of the permanent of the generic symmetric matrix is not generated in degree two in general. For $n = 2, 3, 4, 5$ the apolar ideal is generated in degree 2 with the generators W introduced in Proposition 3.13. We will show starting from $n = 6$ there are generators of degree 3 in the annihilator ideal (Lemma 3.18). Here, for the readers' convenience we summarize the information/observation we have about these examples:

- For $n = 2$ we have

$$X = \begin{pmatrix} a & b \\ b & c \end{pmatrix},$$

So the $\text{perm}(X) = b^2 + ac$. The apolar ideal $I = (C^2, BC, B^2 - 2AC, AB, A^2)$. The corresponding Hilbert sequence is $H = (1, 3, 1)$

- For $n = 2, 3, 4, 5$ the apolar ideal is generated by W in degree 2.
- The Hilbert sequences of the apolar algebra of the permanent of the $n \times n$ generic symmetric matrix is as follows:

Table 2: The Hilbert sequence of the Permanent of the generic symmetric matrix

n=2	1	3	1						
n=3	1	6	6	1					
n=4	1	10	21	10	1				
n=5	1	15	55	55	15	1			
n=6	1	21	120	210	120	21	1		
n=7	1	28	231	630	630	231	28	1	
n=8	1	36	406	1596	2485	1596	406	36	1

The above triangle read by rows $0 \leq k \leq n$ is

$$T(n, k) = \frac{\binom{n}{k}(\binom{n}{k} + 1)}{2}.$$

The n -th row sum is $\frac{\binom{2n}{n} + 2^n}{2} = \dim S^s / \text{Ann}(\text{Perm}(X))$.

- We show that for $n = 6, 7, 8$ the apolar ideal has some degree 3 generators:

The terms $y_{i_1 i_2} y_{i_3 i_4} y_{i_5 i_6}$ that appear in the degree 3 polynomials of the apolar ideal for $n = 6, 7, 8$ follow this rule:

$$i_1 \leq i_3 \leq i_5,$$

$$i_1 \leq i_2, i_3 \leq i_4, i_5 \leq i_6.$$

So the terms that appear in the degree three generators of the apolar ideals are exactly the terms that appear in the Hafnians of the 6×6 symmetric sub-matrices.

- For each 6×6 symmetric submatrix, we have five linearly independent homogeneous polynomials (forms) of degree three among the generators. Three of them have six terms and two of them have eight terms. So the number of degree three generators of an $n \times n$ symmetric matrix is equal to $5 \cdot \binom{n}{6}$. We write these five degree-three forms below for $n = 6$ in Lemma 3.17

Lemma 3.18. *Let X be a generic symmetric $n \times n$ matrix. For each symmetric 6×6 submatrix of X we have five minimal generators of degree three in the apolar ideal of the permanent as listed below:*

$$\begin{pmatrix} a & b & c & d & e & f \\ b & g & h & i & j & k \\ c & h & l & m & n & o \\ d & i & m & p & q & r \\ e & j & n & q & s & t \\ f & k & o & r & t & u \end{pmatrix},$$

$$F_1 = EIO - DJO - EHR + CJR + DHT - CIT$$

$$F_2 = DKN - DJO - CKQ + BOQ + CJR - BNR$$

$$F_3 = FIN - DJO - FHQ + BOQ + CJR - BNR + DHT - CIT$$

$$F_4 = EKM - DJO - CKQ + BOQ - EHR + CJR + DHT - BMT$$

$$F_5 = FJM - DJO - FHQ + BOQ + DHT - BMT$$

These $5 \cdot \binom{n}{6}$ polynomials of Y annihilate the permanent of the matrix X .

Proof. We use induction on n . For $n = 6$ it is easy to check that F_1, \dots, F_5 annihilate the permanent of X . Now we assume that for all integer values less than n we have that all the five polynomials coming from the symmetric 6×6 submatrices annihilate the permanent of X . We want to show it for n . Let N be a 6×6 symmetric submatrix of Y , involving the rows and the columns i_1, \dots, i_6 . If a term in the permanent does not contain any of the 15 degree three monomials of the 6×6 Hafnian then F_1, \dots, F_5 annihilate it. Suppose a term in the permanent contains one of the fifteen monomials in the symmetric 6×6 Hafnian. Since these monomials do not appear in any other 6×6 sub matrix of X , by the first

induction step we have shown that these monomials annihilate the permanent of X . These generators are linearly independent mod $(W)_3$, since they involve different variables.

□

Lemma 3.19. *We can write each unacceptable monomial of degree k ($2 \leq k \leq n$), as an explicit element of $S_{k-2}^s W$, where W is the space defined in the Proposition 3.13.*

Proof. We use induction on k . For $k = 2$ the claim is obviously true. We show that the claim is true for $k = 3$. We need to show that the space U_3 of unacceptable monomials in S_3^s are in $S_1^s W$. The unacceptable monomials of degree 3 for the $n \times n$ generic symmetric matrix are of one of the following forms:

- (a) Unacceptable of the form x^2y where x is a diagonal element. The number of these monomials is $n(\frac{n(n+1)}{2})$.
- (b) Unacceptable of the form xyz where x is a diagonal element, $y \neq x$ in the same row or column with x and $z \neq x$, number of these monomials is $n(n-1)(\frac{n(n+1)}{2} - 1)$.
- (c) Unacceptable of the form xyz where x, y, z are non diagonal elements from the same row or column (can be equal to each other), number of these monomials is $\binom{n}{1} \binom{n-1+3-1}{3}$.

Unacceptable monomials of type (a) or (b) are multiples of unacceptable monomials of degree 2, so they are in the space $S_1^s U_2$. So we only need to show that the degree 3 unacceptable monomials of type (c) are in $S_1^s W$. The 3 non diagonal elements in the same row or column of the matrix X are from a symmetric 4×4 sub-matrix. So without loss of generality we show that a degree 3 monomial of type (c) from the following sub-matrix is in $S_1^s W$. Let A^s be the 4×4 symmetric sub-matrix of a generic symmetric $n \times n$ matrix,

$$A^s = \begin{pmatrix} a & b & c & d \\ b & e & f & g \\ c & f & h & i \\ d & g & i & j \end{pmatrix},$$

and D^s be the matrix

$$D^s = \begin{pmatrix} A & B & C & D \\ B & E & F & G \\ C & F & H & I \\ D & G & I & J \end{pmatrix}$$

Monomials of type (c) in S_3^s can have one of the following forms. In each case we prove the claim for one monomial in the given form. The proof for any other monomial is similar to what we show below.

(I) All three non-diagonal variables are distinct. The monomials of the form $\eta_1 = BCD$ is a degree three unacceptable monomial of type (c). We have $AF - BC \in W$ (since it is a minor with one diagonal element), so we have $(AF - BC) \circ \text{perm}(A^s) = 0$. Hence,

$$D(AF - BC) \circ \text{perm}(A^s) = 0.$$

We also know that $DAF \in S_1^s U_2 \subset U_3$ so $DAF \circ \text{perm}(A^s) = 0$. We have $\eta_1 = BCD = -D(AF - BC) \pmod{S_1^s U_2}$. So we have $\eta_1 \in S_1^s W$.

(II) There are two distinct non-diagonal variables. Consider the monomial $\eta_2 = B^2 C$, also of type (c). We have $AF - BC \in w$ (since it is a minor with one diagonal element), so we have $(AF - BC) \circ \text{perm}(A^s) = 0$. Hence,

$$B(AF - BC) \circ \text{perm}(A^s) = 0.$$

We also know that $BAF \in S_1^s U_2 \subset U_3$ so $B^2 C \circ \text{perm}(A^s) = 0$. We have $\eta_2 = B^2 C = -B(AF - BC) \pmod{S_1^s U_2}$. So we have $\eta_2 \in S_1^s W$.

(III) There is only one non-diagonal variable. Consider the monomial $\eta_3 = B^3$, also of type (c). We have $-B^2 + 2AE \in W$ (since it is a diagonal minor with the coefficient 2), so we have $(-B^2 + 2AE) \circ \text{perm}(A^s) = 0$. Hence,

$$B(-B^2 + 2AE) \circ \text{perm}(A^s) = 0.$$

We also know that $BAE \in S_1^s U_2 \subset U_3$ so $B^3 \circ \text{perm}(A^s) = 0$. We have $\eta_3 = B^3 = -B(-B^2 + 2AE) \pmod{S_1^s U_2}$. So we have $\eta_3 \in S_1^s W$.

So the lemma is true for $k = 3$ and we have $U_3 \subset S_1^s W$. Let M denote the subspace of V generated by binomials of type (b) and (c) defined in Proposition 3.13. We have shown that $U_3 \subset S_1^s(U + M)$.

Now assume that $k \geq 4$ and the lemma is true for all integers less than k . We want to show that the claim is true for k . Let $\mu = \mu_1 \mu_2 \dots \mu_k$ be an unacceptable monomial of degree k . We can write μ such that $\mu_2 \dots \mu_k$ is an unacceptable monomial of degree $k - 1$ so we have

$$\mu = \mu_1(\mu_2 \dots \mu_k) \in S_1^s(S_{k-3}^s W) = S_{k-2}^s W.$$

So the lemma is true also for k . □

Definition 3.20. Let H be the ideal of the degree three polynomials mentioned in the Lemma 3.18. Let $W^+ = W + H$ denote the ideal generated by the degree 2 polynomials defined in Proposition 3.13 and the degree 3 polynomials corresponding to the 6×6 symmetric submatrices discussed in the Lemma 3.18.

- The number of $k \times k$ permanents of the $n \times n$ generic symmetric matrix is

$$\frac{1}{2} \binom{n}{k} \cdot \binom{n}{k} + \frac{1}{2} \binom{n}{k},$$

choosing two from a subset of $\binom{n}{k}$ elements.

- Let $\{P_k\}$ be the set of all $k \times k$ permanents of X .
- As in the determinant case, let A_k be the set of acceptable monomials in S_k^s .
- Let $\iota : R^s \rightarrow S^s$, $\iota(x_{ij}) = y_{ij}$, and E_k be the subset of A_k defined by

$$\{\iota(\mu) | \mu \text{ an initial monomial (in Lex. order) of some element of } \{P_k\}\}.$$

- Let $E'_k = A_k \setminus E_k$ be the complementary set to E_k in A_k .
- For each $\mu \in A_k$, let $A_{>\mu}$, denote the subset of elements $\nu \in A_k$, such that $\nu > \mu$ in the lexicographic order of S^s .
- Let $[a_1, \dots, a_k | b_1, \dots, b_k]_p$ be the permanent of the $k \times k$ sub matrix with the rows $\{a_1, \dots, a_k\}$ and the columns $\{b_1, \dots, b_k\}$. Recall that for a monomial

$$m = y_{a_1 b_1} \dots y_{a_k b_k} = (a_1, \dots, a_k | b_1, \dots, b_k),$$

we call a pair (b_i, b_j) , with $i < j$, a reversal pair if $b_i \geq b_j$. The initial term of the $k \times k$ permanent $[a_1, \dots, a_k | b_1, \dots, b_k]_p$ is the term $y_{i_1, j_1} y_{i_2, j_2} \dots y_{i_k, j_k}$ such that $i_1 \leq i_2 \leq \dots \leq i_k$ and $j_1 \leq j_2 \leq \dots \leq j_k$ where $\{i_1, \dots, i_k\} = \{a_1, \dots, a_k\}$ and $\{b_1, \dots, b_k\} = \{j_1, \dots, j_k\}$.

Proposition 3.21. *Each acceptable monomial in E'_k ($3 \leq k \leq n$), is the initial monomial (in the reverse Lex. order) of an element of W_k^+ .*

Proof. We use the induction on k , we start with $k = 3$. Let μ be a degree 3 acceptable monomial which is not the initial term of any 3×3 permanent in the lexicographic order. The acceptable monomials, $x_{i_1 i_2} x_{i_3 i_4} x_{i_5 i_6}$, of degree 3 for the $n \times n$ generic symmetric matrices can be listed as follows:

- (a) All 6 indices are distinct
- (b) There is one repeated index
- (c) There are 2 repeated indices

(d) There are 3 repeated indices

Now we discuss each of the above types separately. We discuss each case for one monomial of the given form. The proof for any other monomial of the given type is similar to what we show.

(a) all 6 indices are distinct $x_{i_1 i_2} x_{i_3 i_4} x_{i_5 i_6}$, $(i_1, i_3, i_5 | i_2, i_4, i_6)$. Without loss of generality we can assume these indices are 1,2,3,4,5,6. In order to have a non-initial monomial of this kind, it is enough to have at least one reversal pair. For example in $(1, 2, 3 | 6, 4, 5)$, $6 \geq 4$ so $(6, 4)$ is a reversal pair.

In the doset minor $(i_1, i_3, i_5 | i_2, i_4, i_6)$, without loss of generality we may assume

$$i_1 < i_3 < i_5.$$

Now assume $i_1 = 1, i_3 = 2, i_5 = 3, i_2 = 4, i_4 = 5$ and $i_6 = 6$ then $x_{14} x_{25} x_{36}$ is the initial term in the corresponding 3×3 permanent using the lexicographic order. So in order to have a non-initial monomial we need to assign to i_2, i_4 and i_6 the numbers 4,5 and 6 but not in the order. So we have at least one reversal pair $(i_j, i_k), (j < k)$ where $j, k \in \{2, 4, 6\}$ such that $i_j \geq i_k$.

Now we look at the corresponding 6×6 symmetric sub-matrix. we have

$$X = \begin{pmatrix} a & b & c & d & e & f \\ b & g & h & i & j & k \\ c & h & l & m & n & o \\ d & i & m & p & q & r \\ e & j & n & q & s & t \\ f & k & o & r & t & u \end{pmatrix},$$

$$Y = \begin{pmatrix} A & B & C & D & E & F \\ B & G & H & I & J & K \\ C & H & L & M & N & O \\ D & I & M & P & Q & R \\ E & J & N & Q & S & T \\ F & K & O & R & T & U \end{pmatrix},$$

We consider a degree three non-initial monomial, the terms coming from 6 distinct rows and columns. A general example of this kind is $(1, 2, 5 | 6, 4, 3)$. But we have:

$$FIN - DJO - FHQ + BOQ + CJR - BNR + DHT - CIT \in H,$$

so given $\mu = FIN$, we have

$$f_\mu = FIN - DJO - FHQ + BOQ + CJR - BNR + DHT - CIT \in W_3^+,$$

where

$$-DJO - FHQ + BOQ + CJR - BNR + DHT - CIT \in A_{>\mu}.$$

(b) There is one repeated index. Without loss of generality we can assume these indices are 1,2,3,4,5, with one of them repeated. In order to have a non-initial example of this kind, it is enough to have 1 reversal pair. For example in $(1, 2, 3|4, 1, 5)$, $4 \geq 1$. We can form a 5×5 symmetric matrix with these rows and columns,

$$X = \begin{pmatrix} a & b & c & d & e \\ b & f & g & h & i \\ c & g & j & k & l \\ d & h & k & m & n \\ e & i & l & n & o \end{pmatrix},$$

$$Y = \begin{pmatrix} A & B & C & D & E \\ B & F & G & H & I \\ C & G & J & K & L \\ D & H & K & M & N \\ E & I & L & N & O \end{pmatrix},$$

Now consider the monomial $\mu = y_{14}y_{21}y_{35} = BDL$ as an acceptable monomial of type (b). we have the minor $AH - BD \in W$. Given $\mu = BDL$, $f_\mu = -L(AH - BD) \in S_1^s W \subset W_3^+$, where $AHL \in A_{>\mu}$.

(c) There are 2 repeated indices, Without loss of generality we can assume these indices are 1,2,3,4, with two of them repeated. In order to have a non-initial example of this kind, it is enough to have 1 reversal pair. For example in $(1, 2, 3|2, 1, 4)$, $2 \geq 1$. We can form a 4×4 symmetric matrix with these rows and columns,

$$X = \begin{pmatrix} a & b & c & d \\ b & e & f & g \\ c & f & h & i \\ d & g & i & j \end{pmatrix},$$

$$Y = \begin{pmatrix} A & B & C & D \\ B & E & F & G \\ C & F & H & I \\ D & G & I & J \end{pmatrix},$$

Now we consider the monomial $\mu = y_{12}y_{21}y_{34} = B^2I$ as an acceptable monomial of type (c). Given $\mu = B^2I$, $f_\mu = I(B^2 - 2AE) \in S_1^s W \subset W_3^+$, where $AEI \in A_{>\mu}$.

(d) There are 3 repeated indices, Without loss of generality we can assume these indices are 1,2,3, all of them repeated. In order to have a non-initial example of this kind, it is enough to have 1 reversal pair. For example in $(1,2,3|3,2,1)$, in the second column of tableau $3 \geq 1$. So we can form a 3×3 symmetric matrix with these rows and columns,

$$X = \begin{pmatrix} a & b & c \\ b & d & e \\ c & e & f \end{pmatrix},$$

$$Y = \begin{pmatrix} A & B & C \\ B & D & E \\ C & E & F \end{pmatrix},$$

Now we look at the monomial $\mu = y_{13}y_{22}y_{31} = C^2D$ as an acceptable monomial of type (c). Given $\mu = C^2D$, $f_\mu = -C(BE - CD) \in S_1^s W \subset W_3^+$, where $BEC \in A_{>\mu}$.

So for $k=3$ the claim is true. Now assume that the Proposition 3.21 is true for all integers less than k . We have to show that the Proposition 3.21 is also true for k . Let $\mu = \mu_1\mu_2\ldots\mu_k$ be a degree k acceptable non-initial monomial, we can write μ such that $\mu_2\ldots\mu_k$ is a degree $k-1$ acceptable non-initial monomial (it is enough that the monomial $\mu_2\ldots\mu_k$ includes one reversal). Then by the induction assumption $\mu_2\ldots\mu_k$ is the initial monomial (in rev. lex.) of an element of W_{k-1}^+ . So we have $\mu = \mu_1(\mu_2\ldots\mu_k)$ as the initial monomial (in rev. lex.) of an element of W_k^+ . This completes the proof. \square

Corollary 3.22. *For $2 \leq k \leq n$ we have*

$$(W^+)_k = \text{Ann}(\text{perm}(X)) \cap S_k^s$$

Proof. We have

$$(A) \quad W^+ \circ \text{perm}(X) = 0 \iff W^+ \circ S_{n-2}^s \circ (\text{perm}(X)) = 0 \iff W^+ \circ P_2(X) = 0.$$

(B) $(\text{Ann}(\text{perm}(X))) \cap S_2^s = (W^+)_2$ (By Lemma 3.16) $\Rightarrow S_{k-2}^s(W^+) \circ (S_{n-k}^s \circ \text{perm}(X)) = 0$.
 $\Rightarrow S_{k-2}^s(W^+) \circ P_k(X) = 0$.
 $\Rightarrow (W^+)_k \circ P_k(X) = 0$. (By Remark 1.2)

Therefore

$$(W^+)_k \subset \text{Ann}(P_k(X)) \cap S_k^s.$$

By Remark 1.2 and Lemma 3.15 we have

$$(\text{Ann}(\text{perm}(X)))_k = (\text{Ann}(S_{n-k}^s \circ (\text{perm}(X))))_k = (\text{Ann}(P_k(X)))_k$$

So we have

$$\dim W_k^+ \leq \dim(\text{Ann}(\text{perm}(X)) \cap S_k^s) = \dim S_k^s - \dim P_k(X).$$

On the other hand by the Definition 3.19 the sets U_k and D'_k are linearly independent and form a basis for the corresponding subspaces. So by Lemma 3.18 and Proposition 3.20, we have

$$\dim(W^+)_k \geq \dim S_k^s - \dim P_k(X) = \dim \langle D'_k \rangle + \dim \langle U_k \rangle.$$

So we have

$$\dim(W^+)_k = \dim S_k^s - \dim P_k(X) = \dim(\text{Ann}(\text{perm}(X)) \cap S_k^s).$$

□

Theorem 3.23. *Let X be a generic symmetric $n \times n$ matrix. Then the apolar ideal $\text{Ann}(\text{perm}(X))$ is the ideal W^+ generated in degrees two and three.*

Proof. This follows directly from Proposition 3.20 and Corollary 3.21. □

4 Application to the ranks of the determinant and permanent of the generic symmetric matrix

Let $F \in R^s = \mathbb{k}[x_{ij}]$ be a homogeneous form. The apolarity action of $S^s = \mathbb{k}[y_{ij}]$ on R^s , defines S^s as a natural coordinate ring on the projective space $\mathbf{P}(R_1^s)$ of 1-dimensional subspaces of R_1^s . A finite subscheme $\Gamma \subset \mathbf{P}(R_1^s)$ is apolar to F if the homogeneous ideal $I_\Gamma \subset S^s$ is contained in $\text{Ann}(F)$ ([IK],[RS]).

Definition 4.1. We have the following ranks ([IK] Def. 5.66 , [BR] and [RS])

a. the cactus rank (scheme rank) $cr(F)$:

$$cr(F) = \min\{\deg \Gamma | \Gamma \subset \mathbf{P}(T_1), \dim \Gamma = 0, I_\Gamma \subset \text{Ann}(F)\}.$$

b. the smoothable rank $sr(F)$:

$$sr(F) = \min\{\deg \Gamma | \Gamma \subset \mathbf{P}(T_1) \text{ smoothable}, \dim \Gamma = 0, I_\Gamma \subset \text{Ann}(F)\}.$$

c. the rank $r(F)$:

$$r(F) = \min\{\deg \Gamma | \Gamma \subset \mathbf{P}(T_1) \text{ smooth}, \dim \Gamma = 0, I_\Gamma \subset \text{Ann}(F)\}.$$

d. the differential rank (Sylvester's catalecticant or apolarity bound):

$$l_{diff}(F) = \max\{H(S^s/\text{Ann}(F))_i\}.$$

Proposition 4.2. ([IK], Proposition 6.7C) *The above ranks satisfy*

$$l_{diff}(F) \leq cr(F) \leq sr(F) \leq r(F).$$

Proposition 4.3. (Ranestad-Schreyer) *If the ideal of $\text{Ann}(F)$ is generated in degree d and $\Gamma \subset \mathbf{P}(T_1)$ is a finite (punctual) apolar subscheme to F , then*

$$\deg \Gamma \geq \frac{1}{d} \deg(\text{Ann}(F)),$$

where $\deg(\text{Ann}(F)) = \dim(S^s/\text{Ann}(F))$ is the length of the 0-dimensional scheme defined by $\text{Ann}(F)$.

Using the Ranestad-Schreyer Proposition and our results, we have

Theorem 4.4. *For the determinant of a generic symmetric $n \times n$ matrix X , we have*

$$\frac{1}{2(n+2)} \binom{2n+2}{n+1} \leq cr(\det(X)) \leq sr(\det(X)) \leq r(\det(X)).$$

Theorem 4.5. *For the permanent of a generic symmetric $n \times n$ matrix X , we have*

$$\frac{\binom{2n}{n} + 2^n}{6} \leq cr(\text{perm}(X)) \leq sr(\text{perm}(X)) \leq r(\text{Perm}(X)).$$

Notation. [LT] Let $\Phi \in S^d \mathbb{C}^n$ be a polynomial, we can polarize Φ and consider it as a multilinear form $\tilde{\Phi}$ where $\Phi(x) = \tilde{\Phi}(x, \dots, x)$ and consider the linear map $\Phi_{s,d-s} : S^s \mathbb{C}^{n*} \rightarrow S^{d-s} \mathbb{C}^n$, where $\Phi_{s,d-s}(x_1, \dots, x_s)(y_1, \dots, y_{d-s}) = \tilde{\Phi}(x_1, \dots, x_s, y_1, \dots, y_{d-s})$. Define

$$\text{Zeros}(\Phi) = \{[x] \in \mathbb{P} \mathbb{C}^{n*} \mid \Phi(x) = 0\} \subset \mathbb{P} \mathbb{C}^{n*}.$$

Let x_1, \dots, x_n be linear coordinates on \mathbb{C}^{n*} and define

$$\Sigma_s(\Phi) := \{[x] \in \text{Zeros}(\Phi) \mid \frac{\partial^I \Phi}{\partial x^I}(x) = 0, \forall I, \text{ such that } |I| \leq s\}.$$

In our notation $\Phi_{s,d-s}$ is the map from $S_s \rightarrow R_{n-s}$ taking h to $h \circ \Phi$, hence its rank is $H(\mathfrak{A}_A)_s$.

In the following theorem we use the convention that $\dim \emptyset = -1$.

Theorem 4.6. (Landsberg-Teitler) ([LT]) *Let $\Phi \in S^d \mathbb{C}^n$, Let $1 \leq s \leq d$. Then*

$$\text{rank}(\Phi) \geq \text{rank} \Phi_{s,d-s} + \dim \Sigma_s(\Phi) + 1.$$

Remark. (Z. Teitler) If we define $\Sigma_s(\Phi)$ to be a subset of affine, rather than projective space then the above theorem would not need $+1$ at the end, and would not need the statement that the dimension of the empty set is -1 .

Using the Landsberg-Teitler formula we have:

Proposition 4.7. *Let X be a generic symmetric $n \times n$ matrix. For each t , $1 \leq t \leq n$, we have*

$$r(\det(X)) \geq \frac{\binom{n+1}{t} \binom{n+1}{t+1}}{n+1} + \frac{(n-t-1)(n-t)}{2} + (t+1)(n-t-1) + 1.$$

The maximum of the right hand side of the above inequality occurs at $t = \lfloor n/2 \rfloor$.

Proof. By Lemma 2.5 the dimension of the space of $t \times t$ is the Narayana number $\frac{\binom{n+1}{t}\binom{n+1}{t+1}}{n+1}$. The determinant of X vanishes to order $t+1$ if and only if every minor of X of size $n-t$ vanishes. Thus $\Sigma_t(\det_n)$ is the locus of matrices of rank at most $n-t-1$ so the $\dim \Sigma_t(\det_n)$ is $\frac{(n-t-1)(n-t)}{2} + (t+1)(n-t-1)$ ([BH], page 304). By unimodality of the binomial coefficients the first term of the right hand side is maximum at $t = \lfloor n/2 \rfloor$. \square

Proposition 4.8. (a) ([CCG], page 4) For the monomial $x_1^{b_1} \dots x_n^{b_n}$, where $1 \leq b_1 \leq \dots \leq b_n$ we have

$$r(x_1^{b_1} \dots x_n^{b_n}) = \prod_{i=2}^n (b_i + 1)$$

(b) ([RS], page 2) For the monomial $x_1^{b_1} \dots x_n^{b_n}$, where $1 \leq b_1 \leq \dots \leq b_n$ we have

$$sr(x_1^{b_1} \dots x_n^{b_n}) = cr(x_1^{b_1} \dots x_n^{b_n}) = \prod_{i=1}^{i=n-1} (b_i + 1)$$

Proposition 4.9. ([BR], Theorem 1) Let $F \in R^s$ be a homogeneous form of degree d , and let l be any linear form in S_1^s . Let F_l be a dehomogenization of F with respect to l . Denote by $\text{Diff}(F)$ the subspace of S^s generated by the partials of F of all orders. Then

$$cr(F) \leq \dim_k \text{Diff}(F_l)$$

We thank Pedro Marques for pointing out that it is easy to show that the length of a polynomial is an upper bound for the length of any dehomogenization of that polynomial. so we have the upper bound equal the length for the cactus rank of each polynomial. So we have

$$cr(F) \leq \dim_k \text{Diff}(F) = \deg(\text{Ann}(F))$$

Example 4.10. Let X be a 4×4 generic symmetric matrix. The Hilbert sequence will be $H = (1, 10, 20, 10, 1)$. Now using the Ranestad-Schreyer Proposition 4.3 we have:

$$\deg \Gamma \geq \frac{1}{d} \deg(\text{Ann}(\det(A))) = \frac{1}{2}(42) = 21.$$

Now using the Proposition 4.7 we have

$$r(\det_4) \geq \frac{\binom{4+1}{2}\binom{4+1}{2+1}}{4+1} + \frac{(4-2-1)(4-2)}{2} + (2+1)(4-2-1) + 1 = 25,$$

which is a better lower bound.

The following table gives the lower bounds for the rank and cactus rank of the determinant of an $n \times n$ generic symmetric matrix X , for $2 \leq n \leq 6$, and also for $n \gg 0$ using the Stirling formula. Asymptotically the RS lower bound is $\cong 2^{n+1}$ times that from LT.

Table 3: The determinant of the generic symmetric matrix						
n	2	3	4	5	6	$n \gg 0$
lower bound for $cr(\det(X))$ using RS	2.5	7	21	66	209.5	$\frac{2^{2n+1}}{(n+1)\sqrt{(n+1)\pi}}$
lower bound for $r(\det(X))$ using LT	4	7	25	56	187	$2^n/n\sqrt{n\pi}$
$l_{diff}(\det_n)$	3	6	20	50	175	$\left(\binom{n}{\lfloor n/2 \rfloor}\right)^2 / \lfloor n/2 \rfloor$

The following table gives the lower bounds (LB) for the cactus rank of the permanent of an $n \times n$ generic symmetric matrix X , for $2 \leq n \leq 6$, and also for $n \gg 0$ using the Stirling formula.

Table 4: The permanent of the generic symmetric matrix						
n	2	3	4	5	6	$n \gg 0$
lower bound for $cr(\text{perm}(X))$ using RS	1.6	4.6	14.3	47.3	164.6	$\frac{2^n(2^n + \sqrt{\pi n})}{6\sqrt{\pi n}}$
$l_{diff}(\text{perm}(X))$	3	6	21	55	210	$\frac{\sqrt{2}}{\sqrt{\pi n}} 2^{n-1}$

Note that for $n \leq 8$ and $n = 10$ the $l_{diff}(\text{perm}(X))$ is a larger lower bound. For $n = 9$ and $n \geq 11$ our result using RS is larger than the $l_{diff}(\text{perm}(X))$.

5 Rank using contraction on the polynomial ring

In this section we use the contraction on the usual polynomial ring. This is an unusual choice and gives an answer that is less regular compared to section three. We first looked at this case, and realized in comparing our calculations with some kindly sent us by A. Conca for the annihilator of the determinant of a symmetric matrix, that they were different. What is happening here is that taking the contraction yields information about writing $\det(X)$ as the sum of divided powers, not usual powers. Hence both the Hilbert function and generators of the apolar ideal are different for contraction versus differentiation. Throughout this section, $S^s = \mathbb{k}[y_{ij}]$ acts on $R^s = \mathbb{k}[x_{ij}]$ by contraction as follows:

$$(y_{ij})^k \circ_{co} (x_{uv})^\ell = \begin{cases} x_{uv}^{\ell-k} & \text{if } (i, j) = (u, v), \\ 0 & \text{otherwise.} \end{cases} \quad (5)$$

This action extends multilinearly to the action of S^s on R^s .

Notation. In order to prevent confusion, we use the following notations in this section.

- We use contraction on usual polynomials (see (5)).
- We use the notation \circ_{co} for the contraction on the usual polynomials. We denote by $\text{Ann}_{co}(F)$ the apolar ideal to the usual polynomial F , using the contraction. We denote by $H_{co}(F)$ the corresponding Hilbert function, and cr_{co} , $l_{diff,co}$, and r_{co} the corresponding ranks.
- We denote by F_{div} the divided power form of polynomial F in the divided power ring \mathcal{D}

Definition 5.1. ([IK], page 267) Let \mathcal{D} be the divided power ring. Recall that for $L = a_1x_1 + \dots + a_rx_r \in \mathcal{D}_1$. The divided power $L^{[j]}$ is defined as

$$L^{[j]} = \sum_{j_1 + \dots + j_r = j} a_1^{j_1} \dots a_r^{j_r} x_1^{[j_1]} \dots x_r^{[j_r]}.$$

The Hilbert function and generators of the apolar ideal of a polynomial are different when we use contraction versus differentiation. This can be seen even with an easy example $(x + y)^2$ which is discussed in Example A. This example explains the difference between the following two cases:

I. *The unusual case:* We consider the polynomial F , in the usual polynomial ring R^s , and find the apolar ideal and the corresponding Hilbert function using contraction.

II. *The usual case:* We consider the polynomial F_{div} , in the divided power ring \mathcal{D} , and find the apolar ideal and the corresponding Hilbert function using contraction; this gives us the same Hilbert function and apolar ideal as when we write the polynomial F in the polynomial ring and use the differentiation, provided the characteristic is at least $\deg(F)$ or zero.

Example A. Let $R = \mathbf{k}[x, y]$, and \mathcal{D} be its corresponding divided power ring. Let $F = (x + y)^2 = x^2 + 2xy + y^2 \in R$

I. The apolar ideal of $F = x^2 + 2xy + y^2 \in R$ using contraction is $I = (xy - 2y^2, x^2 - y^2)$, and the corresponding Hilbert function is $H_{co} = (1, 2, 1)$.

II. The apolar ideal of $F_{div} = 2x^2 + 2xy + 2y^2 \in \mathcal{D}$ using contraction is $I = (x - y, y^3)$, and the corresponding Hilbert function is $H = (1, 1, 1)$.

Example 5.2. Let A^s be the symmetric 2×2 matrix,

$$A^s = \begin{pmatrix} x & y \\ y & z \end{pmatrix}.$$

Let

$$D^s = \begin{pmatrix} X & Y \\ Y & Z \end{pmatrix}.$$

Let $S^s = \mathbf{k}[X, Y, Z]$ act on $R^s = \mathbf{k}[x, y, z]$ by contraction. Let

$$f = \det(A^s) = xz - y^2$$

and let $G = XZ + Y^2$ be the permanent of D^s . Then we have

$$G \circ_{co} f = (XZ + Y^2) \circ_{co} (xz - y^2) = 1 - 1 = 0.$$

Thus $G \in \text{Ann}_{co}(f)$.

In this case the number of 1×1 linearly independent minors are the number of variables which is 3. Hence the Hilbert sequence corresponding to that is $H_{co} = (1, 3, 1)$. Now using the Ranestad-Schreyer Proposition we have:

$$cr_{co}(\det(A^s)) \geq \frac{1}{d} \deg(\text{Ann}_{co}(\det(A^s))) = \frac{1}{2}(5).$$

Here the lower bound given by Ranestad-Schreyer Proposition for $cr_{co}(\det(A^s))$ is the same as the lower bound for $cr(\det(A^s))$ (see Table (3)).

In this example $l_{diff}(\det(A^s)) = 3$.

On the other hand we have

$$f = \det(A^s) = xz - y^2 = 1/2(x+z)^{[2]} - 1/2(x-z)^{[2]} - 1/2y^{[2]}.$$

so we have

$$l_{diff,co}(\det(A^s)) = 3 \leq r(\det(A^s)) \leq 3.$$

So by Proposition 4.2 we have

$$r_{co}(\det(A^s)) = sr_{co}(\det(A^s)) = cr_{co}(\det(A^s)) = l_{diff,co}(\det(A^s)) = 3.$$

Example 5.3. Let A^s be a symmetric 3×3 matrix,

$$A^s = \begin{pmatrix} x & y & u \\ y & z & v \\ u & v & w \end{pmatrix}.$$

Let

$$D^s = \begin{pmatrix} X & Y & U \\ Y & Z & V \\ U & V & W \end{pmatrix}.$$

Here $S^s = \mathbb{k}[X, Y, U, Z, V, W]$ acts on $R^s = \mathbb{k}[x, y, u, z, v, w]$ by contraction. Let $f = \det(A^s) = -u^2z + 2yuv - xv^2 - y^2w + xzw$. Let P_{ii} be the permanent corresponding to the entry d_{ii} of the matrix D^s . We have

$$P_{11} \circ_{co} f = (ZW + V^2) \circ_{co} (x(zw - v^2) - y(yw - uv) + u(yv - zu)) = x - x = 0.$$

Thus $P_{11} \in \text{Ann}_{co}(f)$. It is easy to see that when $n = 3$, $P_{ij} \circ_{co} f = 0$ for each $1 \leq i, j \leq 3$. So in the case $n = 3$ the annihilator of the determinant of a symmetric matrix certainly contains all its principal 2×2 permanents of D^s . Using Macaulay 2 for calculations we have:

$$\begin{aligned} \text{Ann}_{co}(\det(A^s)) = & (W^2, VW, UW, V^2 + ZW, ZV, UV + 2YW, Z^2, \\ & 2UZ + YV, YZ, U^2 + XW, YU + 2XV, XU, Y^2 + XZ, XY, X^2) \end{aligned}$$

In this case the number of 1×1 linearly independent minors are the number of variables which is 6. The number of 2×2 linearly independent minors is also 6. So the Hilbert

sequence corresponding to that is $H_{co} = (1, 6, 6, 1)$, which is the same as what we had in the usual II case (see Table 1). Now using Ranestad-Schreyer Proposition we have:

$$cr_{co}(\det(A^s)) \geq \frac{1}{d} \deg(\text{Ann}_{co}(\det(A^s))) = \frac{1}{2}(14) = 7.$$

Here the lower bound given by Ranestad-Schreyer Proposition for $cr_{co}(\det(A^s))$ is the same as the lower bound for $cr(\det(A^s))$ (see Table (3)).

In this example $l_{diff,co}(\det(A^s)) = 6$.

On the other hand, [LT] table 1, describes the ranks of cubic forms as follows:

$$r_{co}(x^2y) = 3, r_{co}(xyz) = 4$$

So by the expansion of the determinant and the above equations we have

$$7 \leq cr_{co}(\det(A^s)) \leq r_{co}(\det(A^s)) \leq 17$$

By Landsberg-Teitler Proposition a lower bound for the rank is 10.

Example 5.4. When A^s is a 4×4 symmetric matrix,

$$A^s = \begin{pmatrix} a & b & c & d \\ b & e & f & g \\ c & f & h & i \\ d & g & i & j \end{pmatrix}.$$

and

$$D^s = \begin{pmatrix} A & B & C & D \\ B & E & F & G \\ C & F & H & I \\ D & G & I & J \end{pmatrix}.$$

Using Macaulay 2 for calculations, we find that the corresponding Hilbert sequence will be $H_{co} = (1, 10, 33, 10, 1)$. Now using the Ranestad-Schreyer Proposition we have:

$$cr_{co}(\det(A^s)) \geq \frac{1}{d} \deg(\text{Ann}_{co}(\det(A^s))) = \frac{1}{3}(55) = 18.33.$$

This is the first example where the lower bound given by Ranestad-Schreyer Proposition for $cr_{co}(\det(A^s))$ is smaller than the lower bound for $cr(\det(A^s))$ (see Table (3)).

In this example $l_{diff,co}(\det(A^s)) = 33$ and using the Landsberg-Teitler Proposition a lower bound for the rank of the determinant as a sum of divided powers is 38.

Example 5.5. When A^s is a 5×5 symmetric matrix, the Hilbert sequence will be $H_{co} = (1, 15, 85, 85, 15, 1)$ according to our calculations using Macaulay 2 and the maximum degree of the generators of $\text{Ann}_{co}(\det(A^s))$ is 3. Now using the Ranestad-Schreyer Proposition we have:

$$cr_{co}(\det(A^s)) \geq \frac{1}{d} \deg(\text{Ann}_{co}(\det(A^s))) = \frac{1}{3}(202) = 67.33.$$

Here the lower bound given by Ranestad-Schreyer Proposition for $cr_{co}(\det(A^s))$ is larger than the lower bound for $cr(\det(A^s))$ that is 66 (see Table (3)).

In this example $l_{diff,co}(\det(A^s)) = 85$ and using the Landsberg-Teitler Proposition a lower bound for the rank as a sum of divided powers is 95.

Example 5.6. When A^s is a 6×6 symmetric matrix, the Hilbert sequence will be $H_{co} = (1, 21, 180, 485, 180, 21, 1)$ according to our calculations using Macaulay 2 and the maximum degree of the generators of $\text{Ann}_{co}(\det(A^s))$ is 4. Now using the Ranestad-Schreyer Proposition we have:

$$cr_{co}(\det(A^s)) \geq \frac{1}{d} \deg(\text{Ann}_{co}(\det(A))) = \frac{1}{4}(889) = 222.25.$$

Here the lower bound given by Ranestad-Schreyer Proposition for $cr_{co}(\det(A^s))$ is larger than the lower bound for $cr(\det(A^s))$ that is 209.5 (see Table (3)).

In this example $l_{diff,co}(\det(A)) = 485$ and using the Landsberg-Teitler Proposition a lower bound for the rank as a sum of divided powers is 497.

Example 5.7. When A^s is a 7×7 symmetric matrix, the Hilbert sequence will be $H_{co} = (1, 28, 336, 1505, 1505, 336, 28, 1)$ according to our calculations using Macaulay 2 and the maximum degree of the generators of $\text{Ann}_{co}(\det(A^s))$ is 4. Now using the Ranestad-Schreyer Proposition we have:

$$cr_{co}(\det(A^s)) \geq \frac{1}{d} \deg(\text{Ann}_{co}(\det(A))) = \frac{1}{4}(3740) = 935.$$

Here the lower bound given by Ranestad-Schreyer Proposition for $cr_{co}(\det(A^s))$ is larger than the lower bound for $cr(\det(A^s))$ that is 715.

In this example $l_{diff,co}(\det(A)) = 1505$, and using the Landsberg-Teitler Proposition a lower bound for the rank as a sum of divided powers is 1524.

The next proposition concerns generators for the apolar ideal in the unusual setting of contraction acting on the usual determinant of A^s .

Proposition 5.8. *Let A^s be a generic symmetric $n \times n$ matrix. For $n > 3$, the generators of degree 2 of the ideal $\text{Ann}_{co}(\det(A^s))$ are:*

(a) x^2 , where x is a diagonal element.

(b) xy , where x is a diagonal element and y is in the same row or column as x .

(c) all the diagonal 2×2 permanents.

So we have:

$$\dim_k(\text{Ann}_{co}(\det(A^s)))_2 = n^2 + \binom{n}{2}.$$

Proof. Let $A^s = (x_{ij})$ and $D^s = (y_{ij})$ be symmetric matrices in R^s and S^s respectively, we have $\det(A^s) = \sum_{\sigma \in S_n} \text{Sgn}(\sigma) \Pi x_{i, \sigma(i)}$. First we show that the monomials of type (a) and (b) are in $\text{Ann}(\det(A^s))$. We have

$$\begin{aligned} y_{ii}y_{ij} \circ_{co} \sum_{\sigma \in S_n} \text{Sgn}(\sigma) \Pi x_{i, \sigma(i)} &= 0 \text{ (where } j \geq i) \\ y_{ii}y_{ji} \circ_{co} \sum_{\sigma \in S_n} \text{Sgn}(\sigma) \Pi x_{i, \sigma(i)} &= 0 \text{ (where } j \leq i) \end{aligned}$$

Now we want to show that the binomials of type (c) are in $\text{Ann}(\det(A^s))$. Let

$$P = y_{ii}y_{jj} + y_{ij}^2,$$

be a 2×2 diagonal permanent, Assume that we have an arbitrary 2×2 diagonal permanent $P = y_{ii}y_{jj} + y_{ij}^2$.

$$P = \text{perm} \begin{pmatrix} y_{ii} & y_{ij} \\ y_{ij} & y_{jj} \end{pmatrix},$$

$$\det(A) = \sum_{\sigma \in S_n} \text{Sgn}(\sigma) \Pi x_{i, \sigma(i)}.$$

There are $n!$ terms in the expansion of the determinant. If a term doesn't contain the monomial $x_{ii}x_{jj}$ or the monomial x_{ij}^2 then the result of action of the permanent P on it

will be zero. Let σ_1 be a permutation having x_{ii} and x_{jj} respectively, in its i -th and j -th place. Corresponding to σ_1 we also have a permutation $\sigma_2 = \tau\sigma_1$, where $\tau = (i, j)$ is a transposition and $\text{sgn}(\sigma_2) = \text{sgn}(\tau\sigma_1) = -\text{sgn}(\sigma_1)$. Thus corresponding to each positive term in the determinant which contains the monomial $x_{ii}x_{jj}$ or the monomial x_{ij}^2 we have the same term with the negative sign, thus the resulting action of the permanent P on $\det(A^s)$ is zero.

Let V be the space generated by generators of type (a), (b) and (c), we have shown that

$$V \subset \text{Ann}_{co}(\det(A^s))_2 = (\text{Ann}_{co}(S_{n-2}^s \circ_{co} (\det(A^s)))_2. \quad (6)$$

Let $W = V^\perp \subset S_2^s$. The space W is generated by generated by:

- (a') $y_{ij}y_{kl}$, where $i \neq j, k \neq l$ and $(i, j) \neq (k, l)$.
- (b') $y_{ii}y_{jk}$, where $i \neq j, k$.
- (c') $y_{ii}y_{jj} - y_{ij}^2$.

We want to show $W = S_{n-2}^s \circ_{co} (\det A^s)$, we have

$$W \supset S_{n-2}^s \circ_{co} (\det(A^s)).$$

Now we need to show that $W = V^\perp \subset S_{n-2}^s \circ_{co} (\det(A^s))$.

By Laplace's expansion theorem for determinant, we have

$$y_{ii}y_{jj} - y_{ij}^2 \in S_{n-2}^s \circ_{co} (\det(A^s)).$$

It is enough to act by the corresponding cofactor on the determinant to get $y_{ii}y_{jj} - y_{ij}^2$. Next we need to show that monomials of the form (a') are also contained in the ideal $S_{n-2}^s \circ_{co} (\det(A^s))$.

For $y_{ij}y_{kl}$ where i, j, k and l are all distinct without loss of generality we can consider $y_{12}y_{34}$. Since if we act by the permutation $\rho = (1, i)(2, j)(3, k)(4, l) \in S_n$ on $y_{ij}y_{kl}$ we have

$$(1, i)(2, j)(3, k)(4, l) \circ_{co} y_{ij}y_{kl} = y_{12}y_{34}.$$

If we consider the 4×4 case in the Example 5.4 we need to show $BI \in S_2^s \circ_{co} (\det(A^s))$. Which by the expansion of determinant in Example 5.4 is easy to see. We can extend this result to $n \times n$ symmetric matrix.

Now if in (a'), i, j, k and l are not all distinct. Without loss of generality we can consider $y_{12}y_{23}$, since we have:

$$(1, i)(2, j)(3, k) \circ_{co} y_{ij}y_{jk} = y_{12}y_{23}.$$

Now if we consider the 4×4 matrix in the Example 5.4 we need to show $BF \in S_2^s \circ_{co} (\det(A^s))$. We can see this easily by using the expansion of the determinant in the Example 5.4.

It is easy to see that monomials of type (b') are also contained in $S_{n-2}^s \circ_{co} (\det(A^s))$, using the same method as used for (a').

Hence we have

$$\dim_k W = \dim_k V^\perp = \dim_k (S_{n-2} \circ_{co} (\det(A^s))).$$

We also have

$$\dim_k V + \dim_k V^\perp = \dim_k S_2^s.$$

Hence by Equation (6) we have

$$V = \text{Ann}_{co}(\det(A^s))_2.$$

□

Corollary 5.9. *Let $A^s = (x_{ij})$ be a generic symmetric matrix and $D^s = (y_{ij})$ be its corresponding symmetric matrix of differential operators. The only monomials of degree two and three which appear among the minimal generators of $\text{Ann}_{co}(\det(A^s))$ using contraction are:*

- (a) *The monomials of degree two that are the product of a diagonal entry of D^s either with itself or with an entry in same row or column with them.*
- (b) *The monomials of degree three that are the product of three distinct off diagonal entries of D^s which are in the same row or column.*

Proof. From Proposition 5.8 we know that all the degree two monomials of type (a) are in $\text{Ann}_{co}(\det(A^s))$ and these monomials are the only monomials of degree two among the generators. Now we want to show that the monomials of type (b) are contained in $\text{Ann}_{co}(\det(A^s))$. Let $y_{ij}y_{ik}y_{il}$ be an arbitrary monomial of type (b). Since D^s is symmetric we have $y_{ij} = y_{ji}$, $y_{ik} = y_{ki}$ and $y_{il} = y_{li}$. We know that in the expansion of the determinant of A^s we cannot have a term which has two elements from the same row or column, So a monomial term x_{ij} and x_{ki} it cannot contain $x_{il} = x_{li}$. Since $x_{il} = x_{li}$ occurs only for two entries of the matrix, so it should be either in the same row with x_{ij} or in the same column with x_{ki} , which is not possible. Hence we have

$$y_{ij}y_{ik}y_{il} \circ_{co} \sum_{\sigma \in S_n} Sgn(\sigma) \Pi x_{i,\sigma(i)} = 0.$$

Now we show that there is no other monomial among the generators of $\text{Ann}_{co}(\det(A^s))$. Let μ be an arbitrary monomial of degree three or higher not of type (b) but among the generators of $\text{Ann}_{co}(\det(A^s))$, so we should have

$$\mu \circ_{co} \sum_{\sigma \in S_n} Sgn(\sigma) \Pi x_{i,\sigma(i)} = 0$$

If μ is a monomial in $\text{Ann}_{co}(\det(A^s))$ of degree three that is not of type (b) nor a multiple of a monomial of type (a) then we claim μ must have one of the following forms:

- (a') $y_{ij}y_{kl}y_{mn}$, where i, j, k, l, m and n are all distinct.
- (b') $y_{ij}y_{ik}y_{lm}$, where i, j, k, l and m are distinct
- (c') $y_{ii}y_{jj}y_{kk}$, where i, j and k are distinct.
- (d') $y_{ii}y_{jj}y_{kl}$, where i, j, k and l are distinct.
- (e') $y_{ii}y_{jk}y_{jl}$, where i, j, k and l are distinct.
- (f') $y_{ii}y_{jk}y_{lm}$, where i, j, k, l and m are distinct.

Using the expansion of the determinant for $n \geq 4$ it is easy to see that none of the above forms can annihilate the determinant. So we are done. □

Remark 5.10. In the Examples 5.2-5.7, using Macaulay 2 for calculations we have, the Hilbert function for $S^s/\text{Ann}_{co}(\text{Per}(A^s))$ is the same as the Hilbert function of the apolar algebra $S^s/\text{Ann}_{co}(\det(A^s))$. Comparing the degree 2 generators we have:

For $n > 3$, the generators of degree 2 of the ideal $\text{Ann}_{co}(\text{Per}(A^s))$ are:

- (a) x^2 , where x is a diagonal element.
- (b) xy , where x is a diagonal element and y is in the same row or column as x .
- (c) all the diagonal 2×2 minors.

And the only degree 3 monomials that appear among the minimal generators of the apolar algebra $S^s/\text{Ann}_{co}(\text{Per}(A^s))$ are the product of 3 distinct off diagonal entries of D^s which are in the same row or column.

In the following tables we summarize the information about the Hilbert sequence of the apolar algebra of the determinant and permanent of the generic symmetric matrix in the unusual setting, acting by contraction on the polynomial ring. For a comparison to the usual case see Table 1 and 2. Note that in Table 5, polynomial F is the determinant or permanent of the generic symmetric matrix.

Table 5: The Hilbert sequence corresponding to F , unusual contraction

n=2	1	3	1					
n=3	1	6	6	1				
n=4	1	10	33	10	1			
n=5	1	15	85	85	15	1		
n=6	1	21	180	485	180	21	1	
n=7	1	28	336	1505	1505	336	28	1

In the following tables we summarize the information about the lower bound for the length of the determinant and permanent of the generic symmetric matrix as a divided power sum. For a comparison to the usual case see Table 3 and 4. Note that in Table 6, polynomial F is the determinant or permanent of the generic symmetric matrix.

Table 6: The determinant of the generic symmetric matrix, unusual contraction

n	2	3	4	5	6	7
lower bound for $cr_{co}(F)$ using RS	2.5	7	18.33	67.33	222.25	935
lower bound for $r_{co}(F)$ using LT	3	10	38	95	497	1524
$l_{diff,co}(F)$	3	6	33	85	485	1505

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